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CONTENTS

PAGE

CIVIL ENGINEERING

The Continuous Vierendeel Girder, by (Prof I. A. El-Demirdash and H. A. Abdel Wahab)	3
The Energy Loss in some Irrigation Works, by (Prof. Dr. A. Khafagi and Adel M. Kamel)	47
Study of the Jump in a Straight and Divergent Open Channels, by (Prof. Dr. A. Khafagi and Dr. M. Said Abdallah)	83
Minimising the Loss in Head Inside the 'Hydraulic Jump' by using a "Hump", by (Dr. M. S. Abdallah)	137
Modified Sutro-Weir Investigations, by (Aziz Shahwan)	171
Concentrated Force Problems in the Transverse Bending of Elastically Restrained Plates, by (W. A. Bassali and R. H. Dawoud)	205

MECHANICAL ENGINEERING

Rippled Bearings for High Speed Journals, by (Galal S. A. Shawki)	227
Investigation of Heat Transfer to Air in Radial Flow, by (Mahmoud Sobhi Abdel-Salam)	247
Gas-Turbine Performance Under Varying Ambient Conditions, by (N. M. Rafat and S. A. Farag)	261
The Combustion of Fuel Oil Droplets a Historical Review and a Plan for a Future Investigation, by (Rafat N. M. and Mikhail M. I.)	279
Compounding the Normally Aspirated Spark Ignition Engine, by (Dr. Talaat El-Sayed Ahmed Youssef)	289
A Study of the Effect of Gas Medium and Oil Temperature on Lubricating Oil Deterioration in an Apparatus Under Conditions as those Present in Crankcases of Splash Lubricated Internal Combustion Engines, by (H. M. El-Sabilgi)	307
A Proposed Scheme for the Power Control of Constant Speed Gas Turbines, by (N. M. Rafat and S. A. Farag)	321
"Force-Extension Characteristic of an Elastic System Dynamically Loaded by an Impulse", by (Prof. Dr. Ahmed Ezzat)	347

AERO. ENGINEERING

Design of Blade Cascades with Minimum Maximum Velocity, by (<i>M. I. I. Rashed</i>)	365
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ELECTRICAL ENGINEERING

Theory of the Squirrel — Cage Induction Motor Based on M.M.F. Consideration, by (<i>M. Noureldin Abdel-Hamid</i>)	405
Appliction of Series Capacitors with Parallel Saturable Inductances in A. C. Transmis- sion Lines, by (<i>M. Noureldin Abdel-Hamid</i>)	419
Filters Possessing Approximate Gaussian Transfer Function, by (<i>Fuad Surial Atiya</i>)	425

THE CONTINUOUS Vierendeel GIRDER

By C. A. ELSTON, M. S. C. E., M. A. S. E. (M. E. C. E.)

U. S. A. AIR FORCE, M. S. C. E., M. A. S. E. (M. E. C. E.)

INTRODUCTION

Although the problem of the Vierendeel girder has been treated by many authors and several methods have been suggested for the determination of the internal forces and stresses, yet little has been mentioned in the general literature of the subject as to the Vierendeel girder and its point of application in the design of continuous systems for composite design as well as for the solution of indeterminate systems in which the Vierendeel girder is involved.

CIVIL ENGINEERING

The stresses of the girders in the different parts of the Vierendeel girder rather than. For different members are subject to normal forces and bending moments at the same time.

$$F, M, S$$

The deformations $\delta_x = \frac{F_x \cdot L}{A \cdot E}$ due to normal forces are of

the same order as the deformations $\delta_x = \int \frac{M_x \cdot y}{EI} dx$ due to bending moments. However, their relative values vary with the dimensions of the system.

The axial forces N of the Vierendeel girder follow the bending moment diagram of an equivalent simple beam. They depend also on the height h . Therefore, the higher the Vierendeel girder, in regard to its span, the smaller are the contributions δ_x to the deformations. This is analogous to the effect of increasing the height of a plate girder or a truss.

THE CONTINUOUS VIERENDEEL GIRDER

BY

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INTRODUCTION

Although the problem of the Vierendeel girder has been treated by many authors and several methods have been suggested for the determination of the internal forces and stresses, yet little has been mentioned in the technical literature of the deflections of the Vierendeel girder and its joint displacements. However, this problem is important for competitive design as well as for the solution of indeterminate systems in which the Vierendeel girder is involved.

The absence of the diagonals in the different panels renders the Vierendeel girder rather elastic. The different members are subject to normal forces and bending moments at the same time.

The deformations $\delta_N = \sum \frac{N_l N_o S}{A E}$ due to normal forces are of the same order as the deformations $\delta_M = \int \frac{M_l M_o ds}{E I}$ due to bending moments. However, their relative values vary with the dimensions of the system.

The axial forces N of the Vierendeel girder follow the bending moment diagram of an equivalent simple beam. They depend also on the height "h". Therefore, the higher the Vierendeel girder, in regard to its span, the smaller are the contributions δ_N to the deflections. This is analogous to the effect of increasing the height of a plate girder or a truss.

On the other hand, the bending moments M of the Vierendeel girder follow the shearing force diagram of an equivalent simple beam. An increase in the height of the girder, in regard to the span, does not change this diagram but leads to a reduction of the chord sections. Consequently, an increase in the height will increase the contributions δ_{Mc} of the chord members to the deflections. It increases as well the contributions δ_{Mv} of the vertical members whose lengths are equal to the height of the girder. In short, an increase in the height of the Vierendeel girder will increase the effect of the bending moments on the deflections.

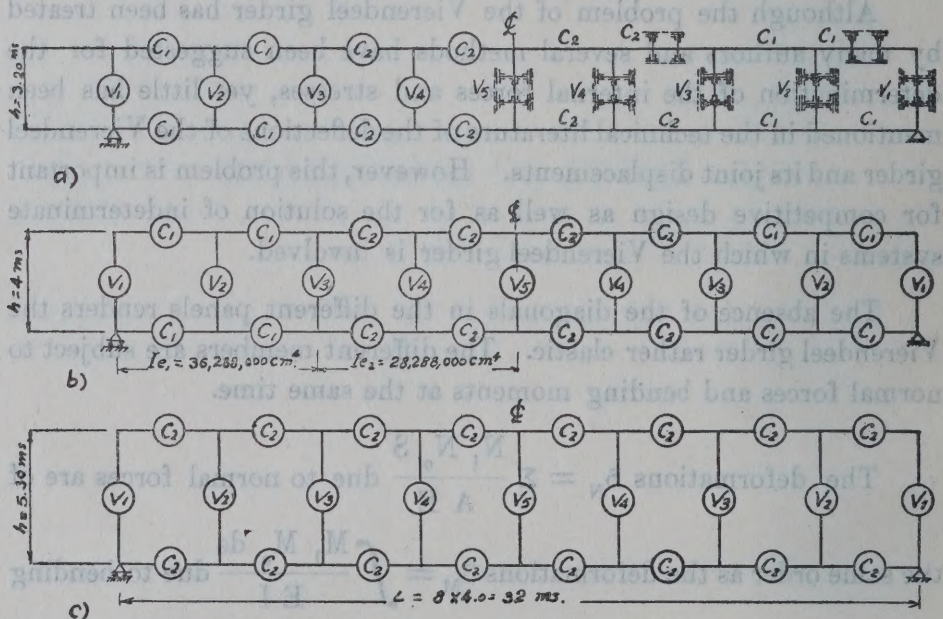


FIG. 1.—Vierendeel Girders a, b, and c

To illustrate the relative effects of the normal forces and bending moments on the deflections of the Vierendeel girder, 3 different girders (Fig. 1) with parallel chords and equal chord stiffness have been investigated. They all have the same span $L = 32$ ms. but heights of 5.3, 4.0 & 3.2 ms. corresponding to $L/6$, $L/8$ and $L/10$ respectively.

FIG 1. (Cont.) VIERENDEEL GIRDERS *a*, *b* & *c*

Section	C_1	C_2	V_1	V_2	V_3	V_4	V_5
	$4 \neq 500 \times 10$	$2 \neq 500 \times 10$	$2 \neq 500 \times 10$	$4 \neq 500 \times 10$	$4 \neq 500 \times 10$	$2 \neq 500 \times 10$	$2 \neq 500 \times 10$
	$8 L^s 100 \times 10$	$8 L^s 100 \times 10$	$4 \neq 250 \times 10$	$4 \neq 250 \times 10$	$4 \neq 250 \times 10$	$4 \neq 250 \times 10$	$1 \neq 400 \times 10$
	$4 \neq 250 \times 10$	$4 \neq 250 \times 10$	$12 L^s 100 \times 10$ $1 \neq 400 \times 10$ $2 \neq 300 \times 10$	$4 \neq 250 \times 12$ $12 L^s 100 \times 10$ $1 \neq 400 \times 10$ $2 \neq 300 \times 10$	$12 L^s 100 \times 10$ $1 \neq 400 \times 10$ $2 \neq 300 \times 10$	$12 L^s 100 \times 10$ $1 \neq 400 \times 10$	$12 L^s 100 \times 10$
$A_g \text{ cm}^2$	453.6	353.6	520.4	750.4	630.4	460.4	360.4
	183, 516	163, 766	169, 314	275, 164	190, 164	164, 814	99, 314

It is interesting to notice that the maximum bending moments occur in the members of the end panels and decrease gradually towards the centre. On the other hand, the normal forces are small at the supports and increase gradually towards the centre of the girder. Their effect on the design of the Vierendeel girder becomes more and more important as the height is decreased.

For this reason, the chord members at the ends of the V. girder may be designed entirely from the corresponding bending moments. However, at the middle of the span the chord stresses due to normal forces are of the same order as those due to bending moments. It is therefore necessary to consider the effects of both normal forces and bending moments in the design of the central chord members.

Referring to Fig. 1, it is noticed that girders (a) and (b) have the same chord sections while girder (c) has a constant chord area. Furthermore, the vertical members in all three girders have the same cross-sections. This is due to the fact that normal forces and bending moments produced in these members are almost independent of the height of the Vierendeel girder.

In the design of all three girders the effect of rigid connections and haunches has been considered. The corresponding stresses at the ends of the haunches as well as midway between these sections have been verified. Due consideration has also been given to the combined stresses. All these considerations were found necessary in order to obtain practical designs.

In order to study the effect of the normal forces and bending moments on the deformations, the deflections δ_N and δ_M at the centre of the span for V. girders (a), (b) and (c), and the cases of loading (i), (ii) and (iii) are given in table (1). They are worked out for the stat. indet. systems as well as for their stat. det. main systems formed by introducing hinges in the mid-points of the different members. The results are then compared with those of a simple beam having a moment of inertia $I_c = 2A_c \left(\frac{h}{2}\right)^2 = \frac{1}{2} A_c h^2$ where A_c is the cross-sectional area of the chord member of the V. girder.

In working out the deflections of the V. girders, the haunches and the stiffness of the joints have not been considered. Since these have a favourable effect on the deformations of the V. girders, the comparative values given in table (1) are higher than the real deflections of the Vierendeel girders.

A study of the results given in table (1) shows the following characteristics :

1. The deflections of the V. girder are bigger than the deflections of the equivalent beam. The latter are almost equal to the contributions δ_N of the normal forces to the deflections of the Vierendeel girder. Thus, connection of the chord members by vertical posts does not provide the same stiffness as connection of the flanges by a full web.

2. For all cases of loading and all girders, the contributions δ_N and δ_M to the deflections of the V. girder are more or less of the same order. In this respect, the V. girder differs from the lattice girder or truss where the effect of bending moments due to rigid connections on the deformations is generally neglected.

3. The contributions δ_M of the bending moments to the deflections of the Vierendeel girder are in this case more important than the contributions δ_N of the normal forces. The higher the girder with respect to its span the bigger will be the ratio $\frac{\delta_M}{\delta_N}$ and subsequently $\frac{\delta_M}{\delta_T}$. On the other hand, the bigger the ratio $\frac{h}{L}$ the smaller will be the ratio $\frac{\delta_N}{\delta_M}$ and subsequently $\frac{\delta_N}{\delta_T}$.

4. The ratio $\frac{\delta_N}{\delta_T}$ is minimum for the case of a single concentrated load at the centre. The same ratio is bigger for the case of an unsymmetrical load at the quarter-point. On the other hand the ratio $\frac{\delta_M}{\delta_T}$ is maximum for a single concentrated load at the centre. The same ratio, however, is smaller for a single load at the quarter-point.

TABLE 1
DEFLECTIONS OF THE VIERENDEEL GIRDERS (FIG. 1)
(t, cm. units)

Case of loading	Def. ⁿ at Centre	(a) $h = \frac{L}{6}$		$E\delta_M'$ of Equivalent Beam	(b) $h = \frac{L}{8}$		$E\delta_M'$ of Equivalent Beam	(c) $h = \frac{L}{10}$		$E\delta_M'$ of Equivalent Beam
		Stat. Ind. V.G.	Stat. Det. V.G.		Stat. Ind. V.G.	Stat. Det. V.G.		Stat. Ind. V.G.	Stat. Det. V.G.	
(i) P = 1 t at mid-span	$E\delta_M$	88.425	90.720	13.700	73.493	74.493	22.438	65.553	65.750	35.000
	$E\delta_N$	13.485	14.085		22.732	23.473		35.098	36.098	
	$E\delta_T$	101.910	104.805		96.225	79.966		100.651	101.848	
	δ_M/δ_N	6.550	6.420		3.230	3.135		1.865	1.819	
	δ_M/δ_T	87 %	86.6 %		76.3 %	76.2 %		64.80 %	64.65 %	
(ii) P = 1 t at quarter span	δ_N/δ_T	13 %	13.4 %		23.7 %	23.8 %		35.20 %	35.35 %	
	$E\delta_M$	42.980	45.120	9.440	35.917	35.185	15.590	31.824	31.020	24.300
	$E\delta_N$	9.220	9.650		15.494	15.859		24.877	24.627	
	$E\delta_T$	52.200	54.770		51.412	51.040		56.701	55.647	
	δ_M/δ_N	4.650	4.670		2.320	2.210		1.276	1.268	
	δ_M/δ_T	82.0 %	82.5 %		69.75 %	69 %		56.1 %	56.9 %	
(iii) P = 1 t/m',	δ_N/δ_T	18.0 %	17.5 %		30.25 %	31 %		43.9 %	43.1 %	
	$E\delta_M$	1428.80	1439.30	274.00	1171.520	1194.200	457.600	1038.62	1049.800	710.000
	$E\delta_N$	271.02	272.61		452.814	457.694		703.61	708.882	
	$E\delta_T$	1699.82	1711.91		1624.334	1651.894		1742.23	1759.682	
	δ_M/δ_N	5.28	5.27		2.58	2.610		1.470	1.482	
	δ_M/δ_T	84.2 %	83.9 %		72.1 %	72.2 %		59.5 %	59.75 %	
	δ_N/δ_T	15.8 %	16.1 %		27.9 %	27.8 %		40.5 %	40.25 %	

For both ratios, the values found for the uniformly distributed load lie between the corresponding values for the cases of a single concentrated load $P = 1 \text{ t}$ at the centre and the quarter-point respectively.

5. For all cases of loading and all girders, a remarkable coincidence is established between the deflections of the stat. indet. V. girder and those of its stat. det. main system formed by introducing hinges at the mid-points of the different members. This is true for the total deformations and also valid for the contributions δ_N and δ_M separately.

For this reason, it is recommended, hereafter, to replace the joint displacements of the indeterminate V. girder by those of its stat. det. main system; the maximum error involved in this assumption for all the treated cases of loading and girders is 5 % only. Such a small error is practically of no avail, considering the intricate mathematical computations needed for the determination of the actual deflections of the highly indet. Vierendeel girder.

Furthermore, the errors involved in determining the deflections of the V. girder by this assumption, are smaller than the errors involved in determining the internal forces and moments. This is because the summations needed for the determination of the deflections by virtual work extend over all members. At the same time, the displacements of the actual points of contraflexure, from the mid-points of the different members, are not generally in the same direction. It follows that, if the replacement of the redundant V. girder by its stat. det. main system is not good enough for the determination of the actual normal forces and bending moments, it is however a good assumption for the determination of the deflections.

A comparison is further made between the deflections of the V. girder and the deflections of the plate girder and truss. The three systems investigated are shown in Fig. (2). The results obtained for the deflections, as given in table (2), illustrate the relative stiffness of the different systems.

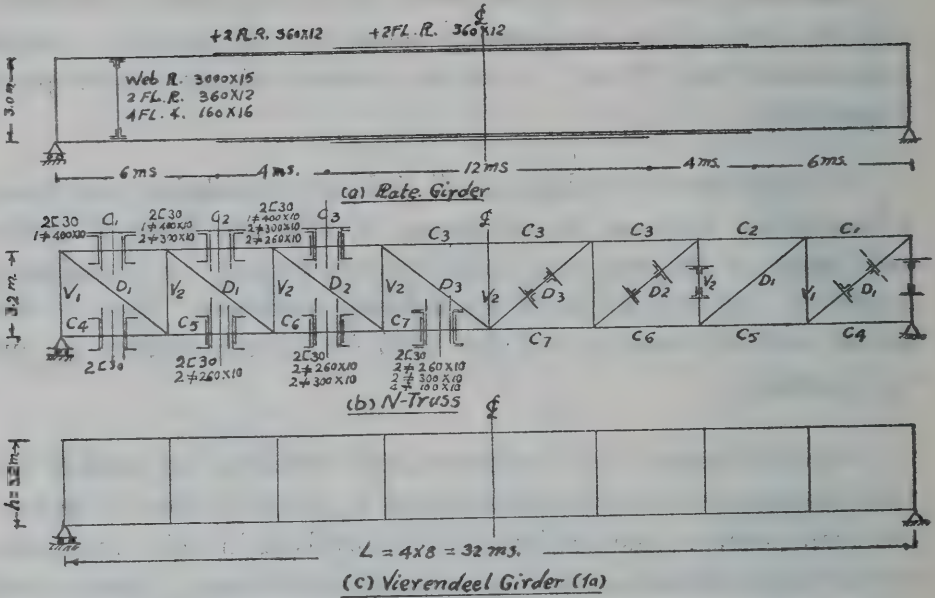


FIG. 2.—Comparative Girders

Furthermore, table (2) shows the following characteristics :

1. The deflections of the V. girder, the N-truss and the plate girder bear roughly the following ratios :

Case of loading	V. Girder	N—Truss	Pl. Girder
Central load	1.00	0.80	0.60
Quarter load:	1.00	0.88	0.68
Distributed load.	1.00	0.83	0.63

This shows that the plate girder is stiffer than the truss and that both systems are stiffer than the Vierendeel girder. The ratio of relative stiffness changes with the loading and reaches its maximum value for a single concentrated load at the centre of the span.

2. The contributions of the chord members of the truss to the deflections are almost equal to the contributions of the bending moments of the plate girder. On the other hand, the contributions of

the web members of the truss to the deflections are considerably bigger than the contributions of the shearing forces of the plate girder. This is obviously due to the effect of the continuous web which gives very small deflections due to shear.

TABLE 2
DEFLECTIONS OF DIFFERENT SYSTEMS
(t, cm. units)

Case of loading	Vierendeel Girder		N-Truss		Plate girder	
$P = 1 \text{ t at mid-span}$	$E \delta_M$	65.553	$E \delta_W$	24.150	$E \delta_Q$	4.440
	$E \delta_N$	35.098	$E \delta_C$	55.000	$E \delta_M$	53.660
	$E \delta_T$	100.651	$E \delta_T$	79.150	$E \delta_T$	58.100
	δ_M/δ_N	1.865	δ_W/δ_C	44%	δ_Q/δ_M	8.34%
	δ_M/δ_T	64.8%	δ_W/δ_T	40.5%	δ_Q/δ_T	7.68%
	δ_N/δ_T	35.2%	δ_C/δ_T	69.5%	δ_M/δ_T	92.32%
$P = 1 \text{ t at } \frac{1}{4} \text{ span}$	$E \delta_M$	31.824	$E \delta_W$	9.700	$E \delta_Q$	3.340
	$E \delta_N$	24.877	$E \delta_C$	40.650	$E \delta_M$	34.970
	$E \delta_T$	56.701	$E \delta_T$	50.350	$E \delta_T$	38.310
	δ_M/δ_N	1.276	δ_W/δ_C	23.85%	δ_Q/δ_M	9.55%
	δ_M/δ_T	56.1%	δ_W/δ_T	19.3%	δ_Q/δ_T	8.7%
	δ_N/δ_T	43.9%	δ_C/δ_T	80.7%	δ_M/δ_T	91.3%
$P = 1 \text{ t/m'}$	$E \delta_M$	1038.620	$E \delta_W$	342.000	$E \delta_Q$	71.000
	$E \delta_N$	703.610	$E \delta_C$	1096.000	$E \delta_M$	1019.700
	$E \delta_T$	1742.430	$E \delta_T$	1438.000	$E \delta_T$	1090.700
	δ_M/δ_N	1.470	δ_W/δ_C	31.25%	δ_Q/δ_M	6.98%
	δ_M/δ_T	59.5%	δ_W/δ_T	23.9%	δ_Q/δ_T	6.51%
	δ_N/δ_T	40.5%	δ_C/δ_T	76.1%	δ_M/δ_T	93.49%

3. The contributions of the normal forces to the deflections of the Vierendeel girder, arise chiefly from the chord members. The posts give negligible values. Furthermore, these contributions are rather smaller than the contributions of the chord members to the deflections of the truss or the contributions of the bending moments to the deflections of the plate girder. This is due to the fact that the chord members of the V. girder are designed not only for the normal forces but also for the co-existing bending moments.

4. The contributions of the bending moments to the deflections of the V. girder are much bigger than the contributions of the web members to the deflections of the truss. Both depend on the shearing forces of the simply supported beam. However, these shearing forces produce axial forces only in the web members of the truss.

CHAPTER I

I.—*The Deformations of the Vierendeel Girder*

The contributions of the normal forces $\delta_N = \Sigma \frac{N_1 N_c S}{A E}$ to the deflections of a V. girder, with equal chord stiffness and constant height, can be calculated from the deflections of an equivalent plate girder whose moment of inertia $I_e = 2 A_c \left(\frac{h}{2}\right)^2 = \frac{1}{2} A_c h^2$.

Referring to the V. girder (I_b) the equivalent plate girder has moments of inertia $I_{e_1} = 36,288,000 \text{ cm}^4$ and $I_{e_2} = 28,288,000 \text{ cm}^4$.

For a concentrated load $P = 1 \text{ t}$ at the centre, the deflection of the equivalent plate girder is $E\delta_{M'} = 22.438 \text{ t/cm}$. This value is nearly the same as the deformation $E\delta_N = 22.732 \text{ t/cm}$. due to normal forces in the V. girder. The small difference between the two values is due to the small contributions of the vertical members of the V. girder.

However, the value $E\delta_N$ is much less than the total deflection $E\delta_T = 96.225 \text{ t/cm}$ given in table (1). It is, therefore, wrong to determine the total deflections of the V. girder by referring the system to an equivalent plate girder.

On the other hand, it is not possible to determine the deflections of the V. girder by considering the effect of the bending moments only. Since the contributions of normal forces cannot be neglected, such a trial leads to erroneous results. Referring to table (1) it is clear, e.g., that for the case of a single concentrated load at mid-span, the contribution $E\delta_M = 73.493 \text{ t/cm}$ is a bad approximation for the total deflection $E\delta_T = 96.225 \text{ t/cm}$.

However, if a reasonable assumption is made for the position of the points of contraflexure the design of the V. girder, and subsequently the calculation of its deflections, can be simplified. According to Saliger, for example, the points of contraflexure for a V. girder with parallel chords under indirect loading are given by the ratios:

$\frac{h_t}{h_b} = \sqrt{\frac{I_t}{I_b}}$ for vertical members and $\frac{a_L}{a_r} = \sqrt{\frac{I_L}{I_r}}$ for chord members as shown in Fig. 3.

For the sake of comparison, the different bending moment diagrams of the V. girder (I_b) for the three cases of loading (i, ii, iii) are drawn in Figs. (4a, 4b, and 4c). The points of contraflexure according to Saliger are shown separately in Fig. (5). These do not depend on the case of loading. The Figs. 4 and 5 show clearly that the assumption of Saliger for the position of the points of contraflexure is not correct in the case of the V. girder.

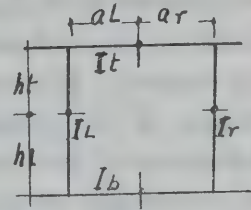


FIG. 3

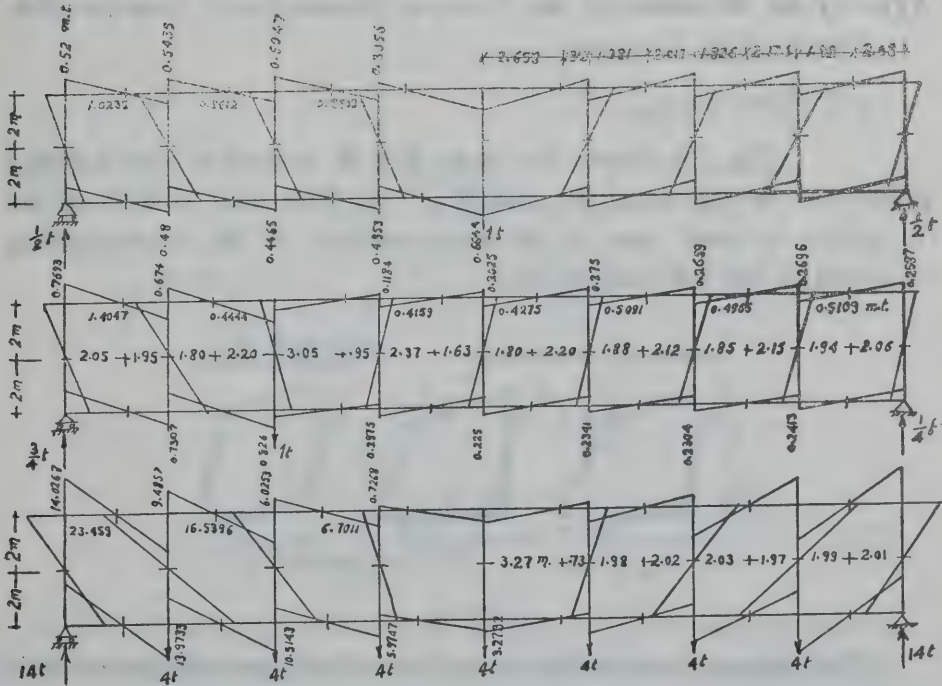


FIG. 4.—B. M. Ds. and Points of Contraflexure for V. Girder (I_b)

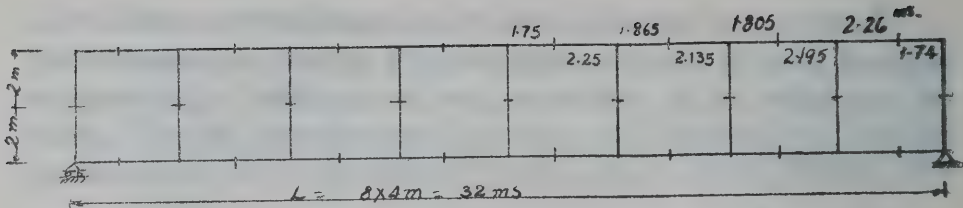


FIG. 5.—Points of Contraflexure according to Saliger

Similarly, the common assumption made for the determination of the stresses in a V. girder with equal chord stiffness, by inserting hinges in the mid-points of its members, gives relatively big errors in the stresses. However, the same assumption leads to a good approximation of the deflections. The results in table (1) show that the deflections of the redundant V. girder are almost identical with those of its statically determinate main system. The position of the point of contraflexure might not always fall together with the mid-point of every member; still the summation for all panels gives a good approximation of the deflections.

Effect of the Deviation of the Points of Contraflexure from the Mid-Points of the Members:

(i) Effect on δ_M

Fig. (6) shows the flow of B.M. around a certain closed panel due to the cases P_0 and $P_1 = 1t$. The total B.M.D. of the V. girder in each case is the superposition of the corresponding diagrams of the different panels.

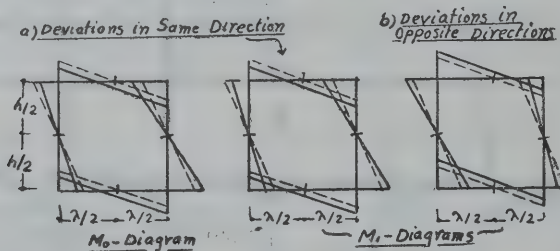


FIG. 6

The points of contraflexure for the verticals are always at their mid-points. On the other hand, the points of contraflexure in the

chord members may deviate from their central mid-points. This deviation is not necessarily in the same direction or to the same extent for the case of deformation P_0 and the virtual loading $P_1 = 1$ t. Calling the B.Ms. of the redundant system M'_0 & M'_1 , and the B. Ms. of the stat. det. main system M_0 & M_1 respectively:

$$\delta_M = \int \frac{M'_1 M'_0 ds}{EI} = \int_c \frac{M'_1 M'_0 ds}{EI_c} + \int_v \frac{M'_1 M'_0 ds}{EI_v},$$

where the indices c and v refer to chord members and verticals respectively.

$$\begin{aligned} \text{But } \int_c \frac{M'_1 M'_0 ds}{EI_c} &= \int \frac{(M_0 + \Delta M_0) (M_1 \pm \Delta M_1) ds}{EI_c} \\ &= \int \frac{M_1 M_0 ds}{EI_c} \pm \int \frac{\Delta M_1 \Delta M_0 ds}{EI_c} \end{aligned}$$

$$\begin{aligned} \text{and } \int_v \frac{M'_1 M'_0 ds}{EI_v} &= \int_{\text{left}} \frac{(M_0 + \Delta M_0) (M_1 + \Delta M_1) ds}{EI_v} \\ &\quad + \int_{\text{right}} \frac{(M_0 - \Delta M_0) (M_1 - \Delta M_1) ds}{EI_v} \\ &\cong 2 \int \frac{M_0 M_1 ds}{EI_v} + 2 \int \frac{\Delta M_1 \Delta M_0 ds}{EI_v} \quad (a) \end{aligned}$$

$$\begin{aligned} \text{or } \int_v \frac{M'_1 M'_0 ds}{EI_v} &= \int_{\text{left}} \frac{(M_0 + \Delta M_0) (M_1 - \Delta M_1) ds}{EI_v} \\ &\quad + \int_{\text{right}} \frac{(M_0 - \Delta M_0) (M_1 + \Delta M_1) ds}{EI_v} \\ &\cong 2 \int \frac{M_0 M_1 ds}{EI_v} - 2 \int \frac{\Delta M_1 \Delta M_0 ds}{EI_v} \quad (b) \end{aligned}$$

In other words, provided that ΔM_1 and ΔM_0 are small quantities, the difference between the contributions δ_M of the exact deflection and that of the hinged system will be a small quantity of the 2nd order.

(ii) Effect on δ_N

Any deviation in the position of the points of contraflexure from the mid-points of each panel gives a corresponding change in the normal forces produced in the chord members. The normal forces in the vertical members, however, are not affected. In other words, any difference in the contributions will arise from the axial forces in the chord members.

Calling the axial forces of the redundant system N'_0 and N'_1 , and the axial forces of the stat. det. main system N_0 and N_1 respectively:

$$\delta_N = \sum \frac{N'_0 N'_1 S}{A E} = \sum_c \frac{N'_0 N'_1 S}{A_c E} + \sum_v \frac{N'_0 N'_1 S}{A_v E}$$

$$\begin{aligned} \text{Hence } \Delta \delta_N &= \sum_c \frac{N'_0 N'_1 S}{A_c E} - \sum_c \frac{N_0 N_1 S}{A_c E} \\ &= \sum_c \frac{\bar{M}'_0 \bar{M}'_1 S}{h^2 E A_c} - \sum_c \frac{\bar{M}_0 \bar{M}_1 S}{h^2 E A_c} \end{aligned}$$

where $\bar{M}'_0$ and $\bar{M}'_1$ are the moments of the simple beam at the points of contraflexure in the redundant system, and $\bar{M}_0$ and $\bar{M}_1$ are the moments at the mid-points of the panels.

Neglecting the small term of the 2nd order :

$$\Delta \delta_N = \frac{S}{E h^2} \sum (\bar{M}_0 \Delta \bar{M}_1 + \bar{M}_1 \Delta \bar{M}_0)$$

This value depends obviously on the the signs of $\Delta \bar{M}_0$ and $\Delta \bar{M}_1$, which need not be the same for all panels and loadings. The contributions of a certain panel may be big but the resultant value of the summation over the whole span is not appreciable (Table 1).

It is further remarked that the differences $\Delta \bar{M}$ are functions of the shearing force (Q) and that, for any case of loading, the shearing force diagram of the simple beam satisfies the condition $\int Q ds = 0$.

Consequently, if all points of contraflexure in the different panels are shifted equally $\Delta \delta_N$ will be zero. In other words, there will be no effect on the deflection of the V. girder due to normal forces.

All these investigations show that the error involved in determining the deflections of the redundant V. girder by reference to its hinged main system lies within the limits of practical approximation. The results given in table (1) support this statement.

II.—The Elastic Line of the V. Girder

Strictly speaking, the deflections of the V. girder are due to the effect of the normal forces, shearing forces and bending moments. Thus $\delta = \delta_N + \delta_Q + \delta_M$. However, the effect δ_Q of the S.F. is generally of no avail. Therefore $\delta = \delta_N + \delta_M$. Furthermore, the deflections of the redundant V. girder are practically the same as those of its stat. det. system formed by introducing hinges in the mid-points of the different members.

(i) *The Contributions δ_N* : The deflection of the V. girder at a certain panel-point due to the effect of the axial forces is given by the expression :

$$\delta_N = \sum_c \frac{N_1 N_0 S}{A_c E} + \sum_v \frac{N_1 N_0 S}{A_v E}$$

Since the virtual case of loading $P = 1$ t stresses three verticals only, the contributions δ_{N_v} will be relatively small. Table 3 gives the values of δ_{N_v} for the V. girder (1b) and cases of loading (i, ii, iii).

Consequently, the effect of the normal forces on the deflections of the V. girder could be approximately limited to the contributions of the chord members. Thus,

$$\delta_N \approx \sum_c \frac{N_0 N_1 S}{A_c E} = \frac{1}{h^2} \sum_c \frac{\bar{M}_0 \bar{M}_1 S}{A_c E}.$$

TABLE 3
COMPARATIVE VALUES OF δ_{N_v} (t, cm. units)

V. Girder	Case (i)	Case (ii)	Case (iii)
E δ_N	22.732	15.495	452.818
E δ_{N_v}	0.373	0.095	3.254
δ_{N_v}/δ_N	1.64%	0.618%	0.719%

This expression is similar to that of an equivalent simple beam (I) whose moment of inertia $I = \frac{1}{2} A_c h^2$. The elastic line of such a beam can be expressed to a good degree of approximation by 2 sine-curves on either side of the maximum ordinate. The truth of this statement has been verified for a simple beam of constant moment of inertia loaded by a single load ($P = 1$ t) at multiples of 1/16th of the span. Table (4) gives the deflections and errors for a single load in the quarter point. In this case, as in all other cases, the deviations of the sine-curves from the actual elastic line are very small.

(ii) *The Contributions δ_M* : The deflection of the V. girder at a certain panel point due to the bending moments is given by the expression: $\delta_M = \int_c \frac{M_1 M_0 ds}{E I_c} + \int_v \frac{M_1 M_0 ds}{E I_v}$

Fig. (7) shows the B.M.D. of the stat. det. hinged system. It can be split into separate B.M.Ds. as shown for panel (III). By equal chord stiffness,

$$M_s = \frac{1}{2} Q_o \cdot S \text{ and } M_c = \frac{1}{4} Q_o \cdot \lambda. \text{ (Fig. 7c)}$$

For the virtual case of loading $P_m = 1$ t, the corner moments for all panels to the left and to the right of m will be

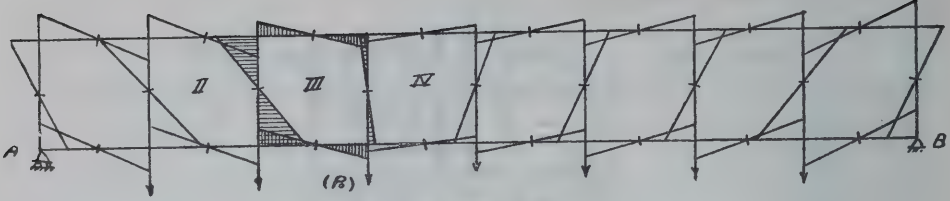
$$\begin{array}{ll} M_c = \frac{1}{4} Q_L \lambda & M_c = \frac{1}{4} Q_r \lambda \\ \text{Left of } m: M_v = \frac{1}{2} Q_L \lambda & \text{Right of } m: M_v = \frac{1}{2} Q_r \lambda \\ M_E = \frac{1}{4} Q_L \lambda & M_E = \frac{1}{4} Q_r \lambda \end{array}$$

TABLE 4
% ERRORS IN THE SINE—CURVE APPROXIMATION
(t, cm. units)

Point	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$E \delta_N$	14.625	27	38.55	48.0	54.6	58.15	59.5	58.67	55.8	51.0	44.70	37.25	28.8	19.65	9.97
$E \delta_{\text{sine}}$	13.10	25.6	36.75	46.3	53.3	57.9	59.5	58.8	56.0	51.8	45.80	38.5	30.0	20.60	10.40
Error (e)	-1.525	-1.40	-1.80	-1.7	-1.3	-0.25	0	0.13	0.2	0.8	1.10	1.25	1.2	1.05	0.43
e/δ %	10.4	5.17	4.65	3.54	2.38	0.43	0	0.222	0.368	1.56	2.46	3.36	4.14	4.33	4.3
e/δ_{max} %	2.57	2.35	3.02	2.86	2.18	0.42	0	0.218	0.337	1.34	1.84	2.10	2.02	1.765	0.72

Area A = 594.02 $A_{\text{sine}} = 592.6$ Error in area = - 1.42 or - 0.238 %

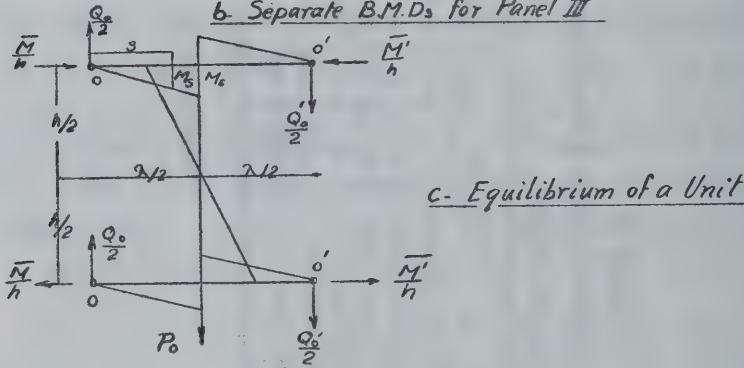
Case of Deformations P_o



a. B.M.D. for Stat. Det. V. Girder.



b. Separate B.M.D.s for Panel III



c. Equilibrium of a Unit

FIG. 7

Fig. (8) shows the B.M.D. for the virtual loading $P_m = 1$. The B.M.D.s. for panel (III) are shown separately in Fig. (8b). Similar to Fig. (7c), the equilibrium of the unit at m is shown in Fig. (8c).

The corresponding contributions to δ_M are

$$\text{Panel at A} \quad : \quad \frac{Q_1 Q_o \lambda}{24 E} \left(\frac{\lambda^2}{I_c} + \frac{\lambda h}{2 I_L} + \frac{\lambda h}{I_r} \right)$$

$$\text{Panel left of m} \quad : \quad \frac{Q_1 Q_o \lambda}{24 E} \left(\frac{\lambda^2}{I_c} + \frac{\lambda h \alpha}{I_L} + \frac{\lambda h}{I_r} \right)$$

$$\text{Panel right of m} \quad : \quad \frac{Q_1 Q_o \lambda}{24 E} \left(\frac{\lambda^2}{I_c} + \frac{\lambda h}{I_L} + \frac{\lambda h \beta}{2 I_r} \right)$$

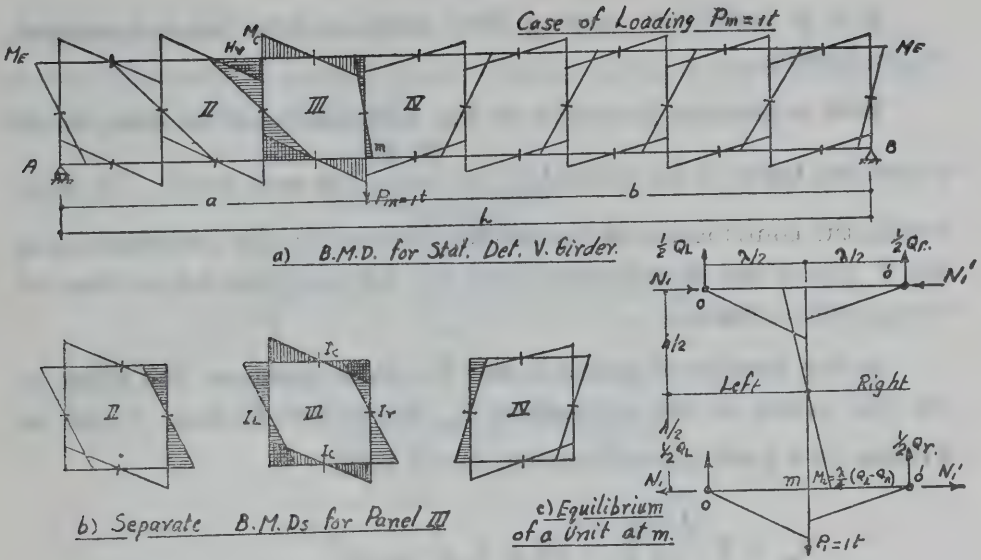


FIG. 8

$$\text{Panel at B} : \frac{Q_1 Q_o \lambda}{24 E} \left(\frac{\lambda^2}{I_c} + \frac{\lambda h}{I_L} + \frac{\lambda h}{2 I_r} \right)$$

$$\text{Intermediate Panel} : \frac{Q_1 Q_o \lambda}{24 E} \left(\frac{\lambda^2}{I_c} + \frac{\lambda h}{I_L} + \frac{\lambda h}{I_r} \right)$$

$$\text{where, } \alpha = \frac{a-b}{a} \text{ and } \beta = \frac{b-a}{b}$$

$$\text{Introducing } r = \frac{I_c}{I_L} \text{ and } s = \frac{I_c}{I_r} :$$

$$\begin{aligned} \delta_M &= \frac{Q_1 Q_o \lambda}{24 E I_c} \left\{ \lambda^2 + \lambda h \left(\frac{r}{2} + s \right) \right\} + \frac{Q_1 Q_o \lambda}{24 E I_c} \left\{ \lambda^2 + \lambda h \right. \\ &\quad \left. \left(r + \frac{s}{2} \right) \right\} + \frac{Q_1 Q_o \lambda}{24 E I_c} \left\{ \lambda^2 + \lambda h \left(r + \frac{\beta}{2} s \right) \right\} \\ &\quad + \frac{Q_1 Q_o \lambda}{24 E I_c} \left\{ \lambda^2 + \lambda h \left(\frac{\alpha}{2} r + s \right) \right\} + \Sigma \frac{Q_1 Q_o \lambda}{24 E I_c} \\ &\quad \left\{ \lambda^2 + \lambda h (r + s) \right\} = \Sigma \frac{Q_1 Q_o \lambda}{24 E I_c} \cdot K. \end{aligned}$$

K is a coefficient varying from panel to panel but independent of the loading.

This expression is similar to the deflections due to shear of an equivalent beam II for which $GA' = \frac{24 EI_c}{K}$ in each panel. In other words, the contribution of the bending moment to the deformations of the V. girder can be calculated from the deformations due to shear of an equivalent beam.

As the number of panels in the V. girder increases the effect of the odd panels in the summation δ_M diminishes so that it can be written, to a good approximation, in the form

$$\delta_M \simeq \Sigma \frac{Q_1 Q_0 \lambda^2}{24 EI_c} [\lambda + h (r + s)]$$

Furthermore, by constant $(r + s)$, the equivalent beam will have a constant cross-section. This is the case, for example, by a constant cross section of the chord members and constant cross section of all verticals. The deflection line of V. girder due to bending moments will be similar to the B.M.D. of the simple beam. For a single concentrated load at one of its panel-points the corresponding deflection line will be a triangle.

In order to investigate how far the previous assumptions are applicable to actual designs, reference has been made to the V. girder (I_b). All cases of loading $P = 1$ t at points 1, 2, 3 and 4 have been investigated. The actual deflections are compared with those of the stat. det. hinged system. Furthermore, the latters are split into their contributions δ_N and δ_M respectively. In this way, it has been possible to compare the δ_N -elastic line of the V. girder with the sine-curves assumed for the deflection of the equivalent beam I, and the δ_M -elastic line with the assumed triangular shape of the deflection due to shear of the equivalent beam II.

Fig. (9) shows the deflection lines for a simple concentrated load at the center of the span. The actual deflections are shown by full straight lines while those referring to the assumption of mid-hinges

are shown dotted. The two lines almost coincide. The maximum errors which occur at point (3) are indicated in the figure. The % age error in area is also given.

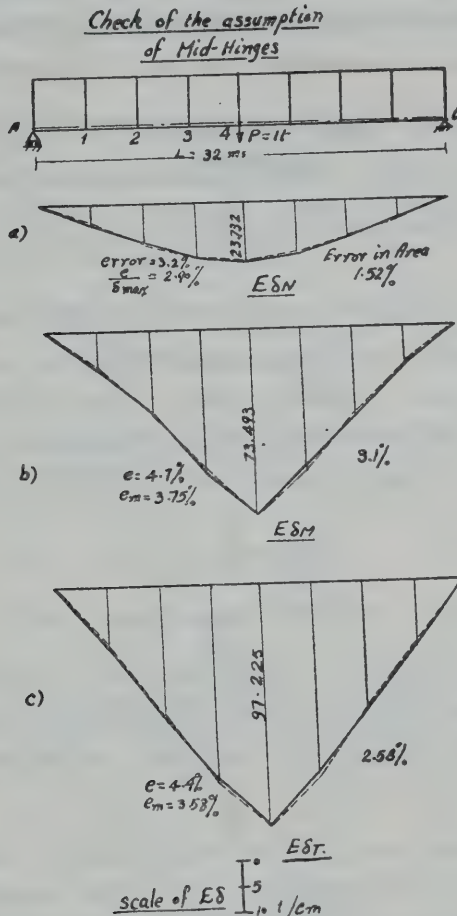


FIG. 9

Fig. (10a) shows the deflections δ_N of the stat. det. V. girder in full lines. The dotted lines refer to the sine-curve deflections of the equivalent simple beam I. The effect of variable moment of inertia is considered in the calculations of the maximum ordinate only. The small error indicated in the ordinate is due to the vertical members of the V. girder. The coincidence of the two lines justifies the proposed assumption.

On the other hand Fig. (10b) shows the deflections δ_M of the stat. det. V. girder. The dotted lines refer to the triangular shape due to shear in the equivalent beam II. The maximum ordinate is here the same as for the stat. det. V. girder. The triangular shape refers to an equivalent beam of constant effective area. The good coincidence justifies the proposed assumption.

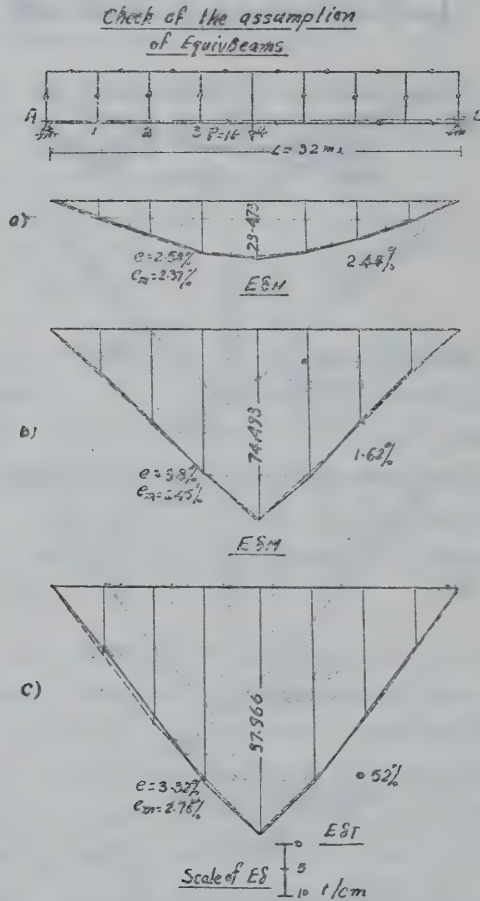


FIG. 10

Finally, the resultant deflections for a single concentrated load $P = 1 \text{ t}$ at the centre of the span are shown in Fig. (10c). The % age errors are indicated in every case. The results obtained for the other cases of panel-point loading have been found to agree fairly well with those obtained by the proposed assumptions.

In short, it is recommended, in working out the deflections, to replace the statically indeterminate Vierendeel girder by the statically determinate main system which is formed by introducing hinges in the mid-points of its members. The total deflections are partly due to normal forces and partly due to bending moments. Thus $\delta = \delta_N + \delta_M$. Both contributions are of the same order.

The values δ_N follow the shape of the deflection line due to the bending moments of an equivalent beam. In most cases, the equivalent beam may be given a constant moment of inertia and the elastic line will be built up of 2 sine-curves.

On the other hand, the values δ_M will follow the shape of the deflection line due to shear of an equivalent simple beam. In most cases, the effective area of the equivalent beam may be assumed constant so that the shape of the deflection line of the V. girder acquires the shape of the bending moment diagram of the actual load.

CHAPTER II

The Continuous Vierendeel Girder

The continuous Vierendeel girder is a highly indeterminate structure. Besides the internal indeterminacy due to the rigid closed panels there is a number of external redundant reactions due to the continuity.

The solution of the continuous V. girder depends largely on the values and influence lines of the redundant reactions. The moments, normal forces and shearing forces of the system are functions of these reactions. In the case of equal chord stiffness, these functions are easily determined by the panel method.

In order to investigate whether the continuous V. Girder gives the same redundant reactions as an equivalent continuous plate girder or truss, the 3 systems (Fig. 11) are treated hereafter. The total length of 32 ms. is divided into 2 equal spans. The plate girder is assumed to have a constant moment of inertia. Further more, all members of the truss are given the same cross section. A similar assumption is also

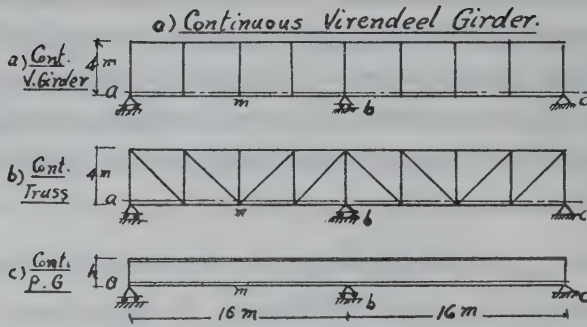


FIG. 11

made for the V. girder. These assumptions are not quite in conformity with practical designs; they serve the purpose of a comparative calculation.

The redundant reactions are calculated for (i) a single concentrated load $P_m = 1$ t at the middle of the left span, and (ii) a uniformly distributed load $p = t/m'$ covering the whole length of the girder. The results obtained for the central redundant reaction in each case are shown in table (5).

A study of these results shows that the reactions of the continuous plate girder do not vary much from those of the continuous truss. However, the values obtained for the 2 systems cannot be adopted for the continuous V. girder. The central reaction for a continuous V. girder is much less than the central reaction for a continuous plate girder or truss.

TABLE 5
REACTIONS R_b FOR THE DIFFERENT SYSTEMS

Case of loading	Continuous V. Girder	Continuous Truss	Continuous Plate Girder
(i) $P_m = 1$ t	0.557 t	0.681 t	0.688 t
	—	+ 22.1 %	+ 22.3 %
(ii) $p = 1$ t/m'	17.280 t	19.000 t	20.000 t
	—	+ 9.95 %	+ 15.75 %

It is, therefore, wrong to take the shearing force and bending moment diagrams of the continuous beam as being valid for the continuous V. girder. The decrease in the value of the central reaction in the case of the continuous V. girder has somehow a similar effect as an imaginary settlement of the central support of the continuous beam.

Furthermore, the deflections of the V. girder are due to the effect of the moments and the normal forces. These 2 values are more or less of the same order. The question now is whether, by neglecting the effect of the moments or that of the normal forces in the deflections, the redundant reactions of the continuous V. girder could be approximately determined. For this purpose, the central reaction of the continuous V. girder, (Fig. 11) has been determined accordingly. The results are contained in table (6). They show that the values obtained for the redundant reaction, by considering the effect of the normal forces only, involve a big %age of error. They are nearly equal to the reactions of the plate girder (Fig. 11c) but cannot be used as good approximation of the real values.

• However, the values obtained by considering the bending moments only, include here a smaller %age of error than usual. This is rather misleading since the ratio $\frac{h}{L} = 1/4$ of Fig. (11a) is relatively high. Consequently, the effect of normal forces is here very small compared with the effect of the bending moments.

TABLE 6

R_b AND %AGE ERRORS FOR CONT. V. GIRDER FIG. (11)

Case of loading	$R_b = -\frac{\delta_{0M}}{\delta_{1M}}$	$R_b = -\frac{\delta_{0N}}{\delta_{1N}}$	$R_b = -\frac{\delta_0}{\delta_1}$
(i) $P_m = 1 \text{ t}$ e	0.52 t -7.2 %	0.70 t +25 %	0.56 t —
(ii) $p = 1 \text{ t/m'}$ e	16.5 t -4.53 %	20.15 t +16.5 %	17.28 t —

In practice, the ratio $\frac{h}{L}$ is generally smaller than $\frac{1}{4}$. Therefore, bigger % age errors will be encountered in determining the reactions if effect of the normal forces is neglected. This is, e.g., the case of the the cont. V. girder (Fig. 12). Table (7) shows the reactions and bigger % age errors for this girder.

TABLE 7
R_b AND % AGE ERRORS FOR CONT. V. GIRDER FIG. (12)

Case of loading	$R_b = \frac{\delta_{0M}}{\delta_{1M}}$	$R_b = \frac{\delta_{0N}}{\delta_{1N}}$	$R_b = \frac{\delta_0}{\delta_1}$
(i) P _m = 1 t	0.545 t	0.705 t	0.630 t
e	-13.5 %	+12.0 %	—
(ii) P = 1 t/m'	32.832	40.448	37.204 t
e	-11.7 %	+8.65 %	—

In short, the effect of both normal forces and bending moments must be considered in the determination of the redundant reactions.

It remains still to investigate the error in the assumption of replacing the continuous V. girder by a continuous hinged system. For this reason, the continuous V. girder (Fig. 12) will be treated hereafter. The total length of 64 ms. is divided into 2 equal spans. Furthermore, the height of the girder is taken $\frac{1}{8} \times 32 = 4$ ms.

A comparison between the sections of the simply supported Vierendeel girder (Fig. 1b) and those obtained for the continuous V. girder (Fig. 12) shows that the V. girder is less affected by continuity than the continuous plate girder or truss.

The contributions of the normal forces and bending moments to the deflection of the simply supported V. girder under a unit load P_b = 1 t are shown in Fig. (13 a). They have been obtained by virtual work from the normal forces and bending moments as found by the panel method. Fig. (13 b) shows the deflections of the hinged system in full lines; the dotted lines refer to the assumptions of the

cine-curve and triangle of the equivalent beams respectively. The coincidence of the dotted lines and full lines in (Fig. 13 *b*) justifies the proposed assumptions. (Fig. 13) also shows that neither the effect of normal forces nor that of the bending moments can be neglected.

Needless to say that the elastic line of the V. girder (Fig. 13 *a*) and that of the hinged V. girder (Fig. 13 *b*) are almost identical.

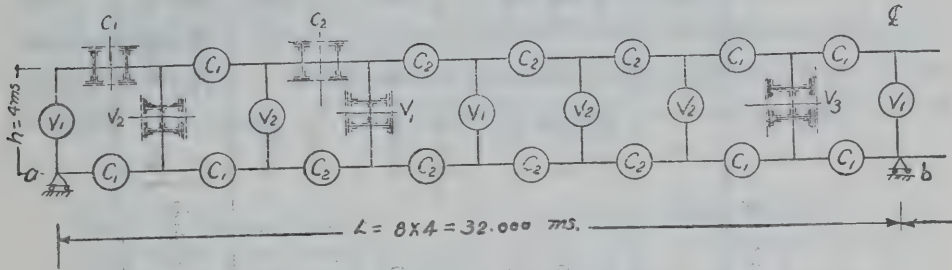


FIG. 12.—Continuous V. Girder

	C ₁	C ₂	V ₁	V ₂	V ₃
Member	4 ≠ 500 × 10	2 ≠ 500 × 10	2 ≠ 250 × 10	4 ≠ 500 × 10	4 ≠ 500 × 10
	4 ≠ 250 × 10	4 ≠ 250 × 10	4 ≠ 250 × 10	8 ≠ 250 × 10	12 ≠ 250 × 10
	8 L ⁸ 100 × 10	8 L ⁸ 100 × 10	1 ≠ 400 × 10	1 ≠ 400 × 10	1 ≠ 400 × 10
			12 L ⁸ 100 × 10	12 L ⁸ 100 × 10	12 L ⁸ 100 × 10
Ag cm ²	453.6	353.6	460.4	670.4	770.4
Ig cm ⁴	183,516	163,766	164,814	256,564	332,364

Therefore, the influence lines of the reaction R_b for the 2 systems are practically the same. Furthermore, this influence line can be obtained from the equivalent beams (Fig. 14). Only the maximum ordinates are here calculated. The part representing the effect of the normal forces is then replaced by a sine-curve while the part representing the effect of the bending moments is replaced by a triangle. The sum of the 2 gives the required influence line of the central reaction.

TABLE 8
INFLUENCE-LINE ORDINATES FOR R_b

Point	1	2	3	4	5	6	7
Rb of Red. System. . .	0.156	0.313	0.468	0.6305	0.750	0.868	0.965
(I) By Mid-Hinges . .	0.155	0.3105	0.470	0.630	0.745	0.868	0.960
(II) By Equiv. Beams . .	0.162	0.321	0.428	0.612	0.735	0.840	0.938
Error e % { I II	-0.640 +3.840	-0.800 +2.550	+0.427 -2.185	-0.079 -2.930	-0.668 -2.000	0 -2.800	-0.518 -2.700
Error e % _m { I II	-0.10 +0.60	-0.25 +0.80	+0.20 -1.00	-0.05 -1.95	-0.50 -1.50	0 -2.80	-0.50 -2.70
Area = 37.204	Approx. Area { I = 37.108 II = 36.628		Error in Area { I = -0.095 or -0.258 % II = -0.512 or -1.376 %				

The errors involved in the different assumptions are contained in table (8). The % age error in the total area in every case is also given. All these errors are relatively very small; therefore, both assumptions are permissible.

However, it is not possible to replace the I.L. of the redundant reaction R_b by the corresponding I.L. of the reaction of continuous beam as found from the effect of bending moments only. This is readily noticed by comparing the relative values of the different deflection lines in (Fig. 13). Such an assumption gives almost a sine-curve, whereas the shape of the I.L. (Fig. 14) is a superposition of

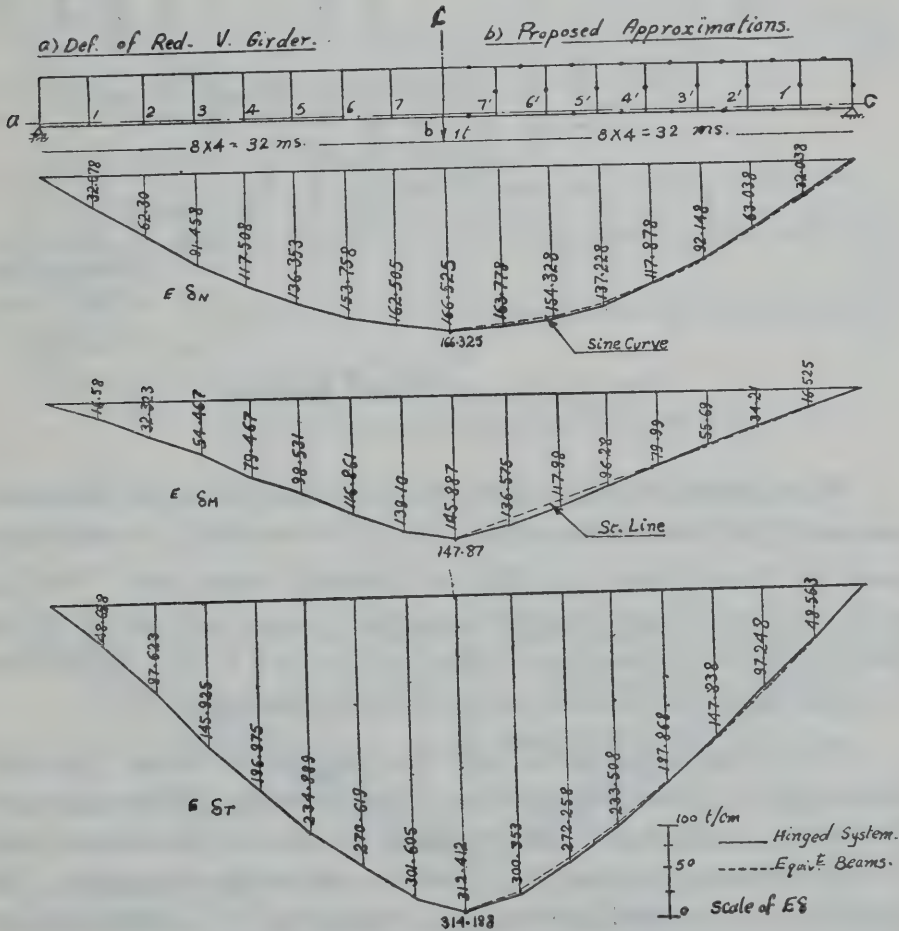


FIG. 13.—Deflections of the V. Girder a-c

a sine-curve and a triangle. Consequently, all errors will be here + ve. How far these errors bear on the result depends on the ratio of (h/L) . The higher the girder with respect to its span the bigger is the error involved in this approximation.

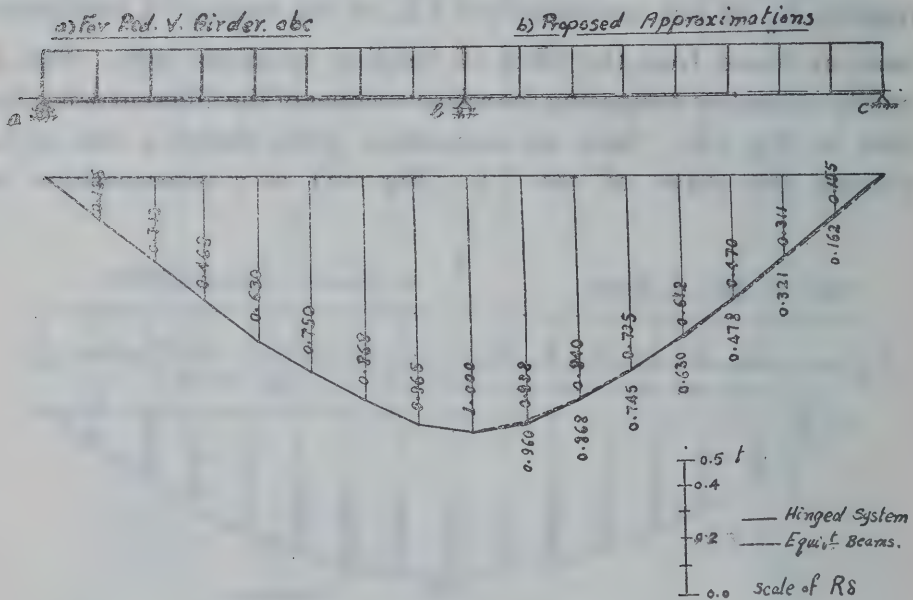


FIG. 14.—I.L. of Central Reaction

Furthermore, it is not allowed to replace the I.L. of the redundant reaction R_b by the corresponding I.L. of the reaction of a continuous beam as found from the effect of shearing forces only. Such an assumption leads to a nearly triangular shape. All errors will be (-ve). Their relative values depend on the ratio h/L . In the case of a shallow V. girder these errors will be considerable.

The foregoing explains why the proposal of Edgar Lightfoot fails to give a good approximation of the deflections of the continuous V. girder. According to this proposal, the effect of the normal forces on the deflections of the V. girder is neglected altogether. The loaded chord is treated as a conjugate beam, Fig. (15), acted upon by the chord moments only.

The end moments of each chord member are not necessarily equal, so that the conjugate beam will have a certain elastic reaction. Besides, the deflections of the conjugate beam will not be the same as those of the actual V. girder. The involved errors in this assumption are shown in (Fig. 16a) where the I.L.s. of the central reaction are drawn.

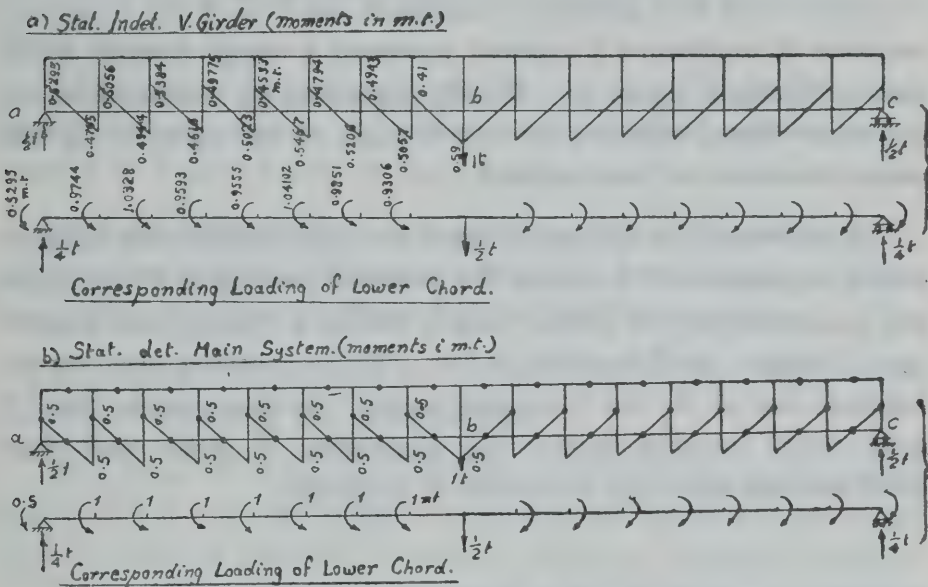
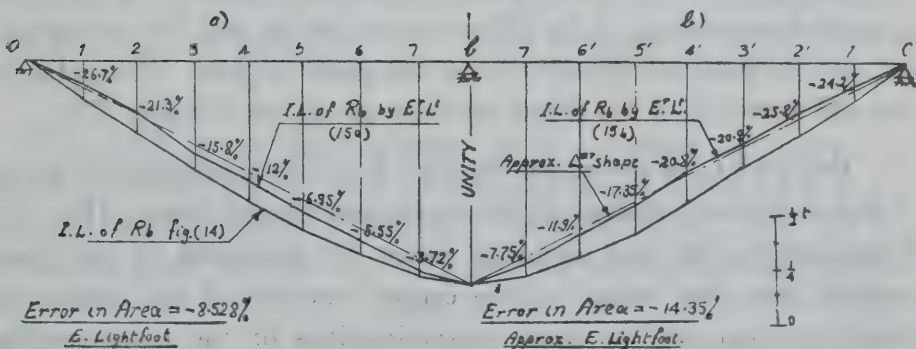


FIG. 15.—Proposed Method of E. Lightfoot.

FIG. 16.— I. Ls. of R_b for Cont. V. Girder (Fig. 12)

E. Lightfoot proposes a further approximation by referring to the stat. det. hinged system. Accordingly, the conjugate beam (Fig. 15b) will have no elastic reactions. Here again, the deflections of the conjugate beam differ considerably from those of the V. girder. The corresponding I.L. with the involved % age errors is shown in Fig. (16b).

A study of the results contained in Fig. (16) shows clearly that the proposals of E. Lightfoot to determine the I.L. of the redundant reactions by continuous V. girders represent a rough attempt which leads to relatively big errors. Besides neglecting the deformations due to normal forces entirely, the contribution of the moments in the vertical members are also omitted.

Furthermore, in the case of constant chord stiffness, the influence line of the reaction R , found by the proposed method of E. Lightfoot and the assumption of middle hinges, will be a triangle with a maximum ordinate equal to unity at (b). This is nothing else but the influence line of R_b for 2 separate spans. In other words, these 2 assumptions together deprive the continuous V. girder with equal chord stiffness from any advantage of continuity.

Influence Lines of Bending Moments and Normal Forces :

Having determined the shape of the I.L. of the central reaction R_b , it is now possible to determine the end moments in the different chord members and verticals of the continuous V. girder. A unit load is applied successively at the different panel-points and the corresponding end moments are determined by the panel method. The I.Ls. of the end moments of the chord members are shown in (Fig. 17).

But the I.L. of the central reaction R_b of the continuous V. girder is almost the same as that of the continuous hinged system (Fig. 14). Consequently, the end moments which are calculated by the panel method from the values of the central reaction of the continuous hinged system will be good approximations of the corresponding moments of the continuous V. girder.

The respective values of the end moments M_{4-5} and M_{5-6} are shown in table 9 (a and b). They have been calculated by the panel method. However, the values of M_h are found from the reaction R_b of the continuous hinged system while the values M_r are found from the reaction of the continuous V. girder. The errors involved in the different ordinates are indicated. Moreover, the percentage errors in the +ve and —ve areas as well as in their algebraic sums are also given.

The results given in table 9 (a and b) show that the bending moment of the continuous V. girder with equal chord stiffness can be obtained to a good degree of approximation in the following way :

1. Assume the continuous V. girder to be provided with hinges in the mid-points of the different members and determine the influence lines of the redundant reactions for this simplified system.

2. Use the obtained I.Ls. for determining the end moments of the cont. V. girder by the panel method.

The I.Ls. of the chord moments of the continuous V. girder shown in Fig. (17) are roughly similar to the influence lines of the shearing forces in a continuous beam. However, the chord moments of the continuous V. girder cannot be calculated from the shearing force values of the continuous beam as if the points of contraflexure lie in the mid-points of the chord members. For the sake of demonstration the I.Ls. of the S.Fs. at the mid-points of the panels 4-5 and 5-6 multiplied by $\frac{\lambda}{4}$ are shown dotted in (Fig. 17). They do not coincide with the actual influence lines of the chord moments of the continuous V. girder.

The I.Ls. of the end moments for the verticals are shown in Fig. (18). They follow, more or less, the shape of the I.Ls. of the chord moments (Fig. 17). The main difference between the two lies in the drop from +ve to —ve. By the end moments of the verticals the drop extends over the 2 panels which lie on either side of the respective vertical. In the case of chord moments, however, the drop occurs in the corresponding panel only.

TABLE 9
(a) I. L. ORDINATES OF $M_{4.5}$

Point	1	2	3	4	5	6	7	B	7'	6'	5'	4'	3'	2'	1'
M_I	0.1277	0.2566	0.3992	0.6589	-0.1923	-0.2656	-0.0947	0	0.0385	0.0486	0.0637	0.0584	0.0410	0.0251	0.0137
M_h	0.1273	0.2558	0.3998	0.6584	-0.1943	-0.2656	-0.0967	0	0.0375	0.0486	0.0627	0.0579	0.0416	0.0247	0.0129
Error	-0.0004	-0.0008	0.0006	-0.0005	0.002	0	0.0020	0	-0.001	0	-0.002	-0.0005	-0.0006	-0.001	-0.0008
e/M %	0.314	0.311	0.205	0.076	1.041	0	2.13	0	2.60	0	3.16	0.852	1.462	3.98	5.9
e/M_m %	0.061	0.121	0.092	0.076	0.304	0	0.304	0	1.517	0	0.303	0.076	0.091	0.152	0.122

$$\begin{aligned}\Sigma A_r &= +4.73958 \\ \Sigma A_h &= +4.66028 \\ e &= -1.68\%\end{aligned}$$

(b) I. L. ORDINATES OF $M_{5.5}$

Point	1	2	3	4	5	6	7	B	7'	6'	5'	4'	3'	2'	1'
M_I	0.1388	0.2878	0.4187	0.5585	0.7835	-0.0958	-0.1313	0	0.0425	0.0601	0.0617	0.0604	0.0446	0.0305	0.0162
M_h	0.1385	0.2875	0.4212	0.5580	0.7825	-0.0958	-0.1323	0	0.0415	0.0601	0.0597	0.0589	0.0456	0.0302	0.0159
Error	-0.0003	-0.0003	0.0025	-0.0005	-0.001	0	0.001	0	-0.001	0	-0.0020	-0.0015	0.001	-0.0003	-0.0003
e/M %	0.228	0.104	0.600	0.090	0.125	0	0.762	0	2.35	0	3.24	2.476	2.235	0.985	1.365
e/M_m %	0.038	0.038	0.319	0.064	0.128	0	0.128	0	0.128	0	0.255	0.255	0.128	0.038	0.023

$$\begin{aligned}\Sigma A_r &= 9.09684 \\ \Sigma A_h &= 9.08196 \\ e &= 0.164\%\end{aligned}$$

$$\begin{aligned}\text{+ve } A_r &= 9.83372 \\ \text{+ve } A_h &= 9.82284 \\ e &= 0.1015\%\end{aligned}$$

$$\begin{aligned}\text{-ve } A_r &= 0.73688 \\ \text{-ve } A_h &= 0.74088 \\ e &= 0.521\%\end{aligned}$$

TABLE 10
(a) I. L. ORDINATES OF N_{4.5}

Point	1	2	3	4	5	6	7	B	7'	6'	5'	4'	3'	2'	1'
N _r	0.374	0.743	1.115	1.470	1.335	0.790	0.495	0	-0.218	-0.350	-0.281	-0.295	-0.243	-0.140	-0.069
N _h	0.376	0.747	1.110	1.475	1.345	0.790	0.510	0	-0.208	-0.350	-0.278	-0.290	-0.248	-0.136	-0.067
Error	0.002	0.004	-0.005	0.005	0.01	0	0.015	0	-0.01	0	-0.003	-0.005	0.005	-0.004	-0.002
e/N %	0.535	0.535	0.45	0.34	0.75	0	3.02	0	4.58	0	1.047	1.70	2.06	3.00	2.90
e/N % _m	0.136	0.272	0.34	0.34	0.68	0	1.02	0	0.68	0	0.204	0.34	0.34	0.272	0.136

$\Sigma A_r = 18.916$
 $\Sigma A_h = 19.104$
 $e = 0.995 \%$

-ve A_r = 6.372
-ve A_h = 6.308
e = -1.005 %

(b) I. L. ORDINATES OF N_{5.6}

Point	1	2	3	4	5	6	7	B	7'	6'	5'	4'	3'	2'	1'
N _r	0.244	0.448	0.6795	0.903	1.17	1.003	0.48	0	-0.255	-0.322	-0.415	-0.347	-0.2555	-0.172	-0.083
N _h	0.245	0.455	0.678	0.908	1.18	1.03	0.49	0	-0.245	-0.322	-0.408	-0.342	0.257	-0.168	-0.082
Error	0.001	0.007	-0.0015	0.005	0.01	0	0.01	0	-0.01	0	-0.008	-0.005	0.0015	-0.004	-0.001
e/N %	0.410	1.56	0.195	0.552	0.852	0	2.08	0	3.9	0	1.69	1.44	0.587	2.33	1.20
e/N % _m	0.086	0.598	0.128	0.476	0.852	0	0.855	0	0.855	0	0.598	0.476	0.128	0.342	0.086

$\Sigma A_r = 12.312$
 $\Sigma A_h = 12.540$
e = 1.86 %

-ve A_r = 7.398
A_h = 7.296
e = 1.39 %

+ve A_r = 19.710
+ve A_h = 19.836
e = 0.638 %

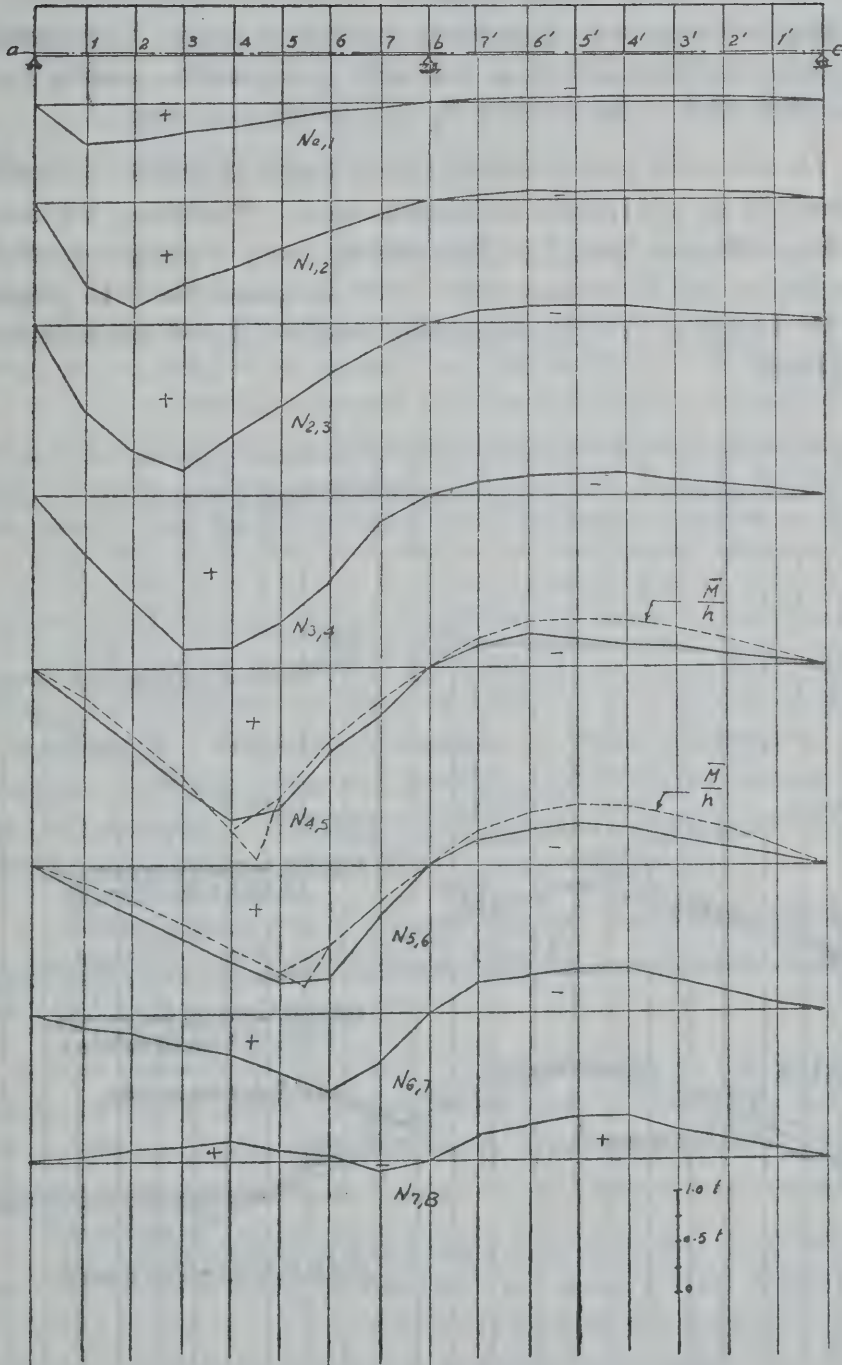


FIG 19.—I.L.s. of Normal forces in Chord Members.

The first contribution, however, is similar to the effect of the bending moments in an equivalent continuous beam. Consequently, the corresponding deflection line. (Fig. 20a) will be similar to the I. L. (Fig. 20c) of the reaction R_b of a continuous beam.

On the other hand the second contribution is similar to the effect of shear in an equivalent continuous beam. Therefore, the corresponding deflection line (Fig. 20b) will be nearly a triangle extending over the 1st and 2nd spans only. This represents the I. L. proposed by E. Lightfoot for the redundant reaction R_b of the continuous V. girder.

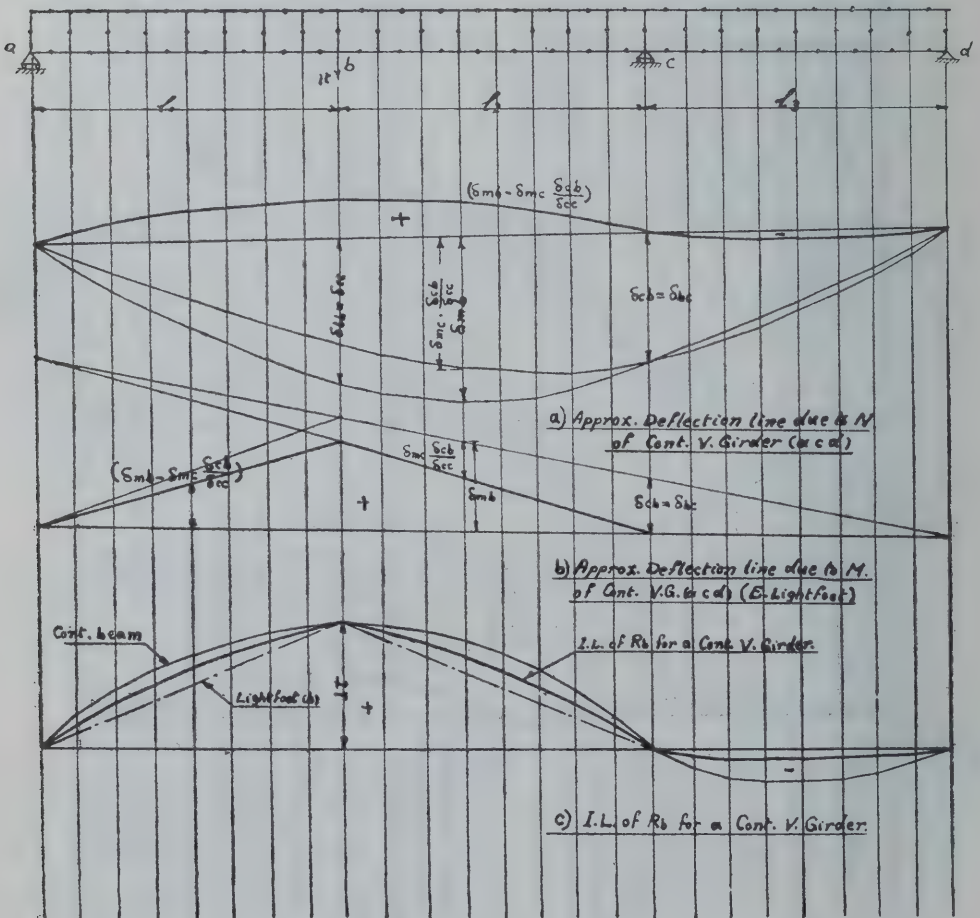


FIG. 20.—Continuous V. Girder over 3 Spans.

It follows that in the case of a continuous V. girder the resultant ordinates in the 3rd span will bear a relatively smaller ratio to the rest of the ordinates of the I. L. R_b than in the case of a continuous beam. Consequently, the V. girder over several spans will be less affected by continuity than a similar continuous beam.

CONCLUSIONS

1. The stresses produced in the chord members of a V. girder are due to the normal forces and bending moments. These follow more or less the bending moments and shearing forces in an equivalent beam. For this reason, the cross-sectional areas of the chord members of the V. girder will not experience the same big changes which are encountered in the flanges or chords of a similar plate girder or truss.

2. The deflection δ of the V. girder is partly due to the contribution " δ_N " of the normal forces and partly due to the contribution " δ_M " of the bending moments. Thus $\delta = \delta_N + \delta_M$. Both contributions are more or less of the same order so that neither of them can be neglected. The ratio between the 2 values depends on the dimensions of the V. girder and especially on the ratio between the height and the span. The higher the girder w. r. t. its span the smaller is the effect δ_N of the normal forces.

3. A plate girder, a truss and a V. girder, designed for the same span and loading, will have different deflections which bear roughly the ratios 1 : 0.8 : 0.6 respectively.

4. The position of the points of contraflexure in the different members of the V. girder depends on the case of loading as well as on the relative stiffness of the members. For this reason, the proposal of Saliger is not practical.

5. The deflections of the internally indeterminate V. girder are almost the same as those of the stat. det. main system formed by introducing hinges in the mid-points of the different members. The coincidence between the 2 values is so good that it renders the determination of the accurate values practically unnecessary.

6. The contributions δ_N of the normal forces to the deflections of the V. girder can be obtained from the deflections due to bending moments of an equivalent beam of variable moment of inertia $I_c = A_c \frac{h^2}{2}$. The corresponding elastic line acquires very nearly the shape of a sine-curve.

On the other hand the contributions δ_M of the bending moments to the deflections of the V. girder can be found from the deflections due to shearing forces of an equivalent beam of variable effective area $A' = \frac{24 EI_c}{G K}$. The corresponding elastic line acquires very nearly the shape of the bending moment diagram of a simple beam under the same external loading.

7. Owing to the flexibility of the V. girder w.r.t. a truss or a plate girder, the redundant reactions of the continuous V. girder will be relatively smaller than those of a continuous truss or plate girder.

8. In determining the redundant reactions of a continuous V. girder it is necessary to include the effect of both normal forces and moments on the deflections. More or less, considerable errors are involved in the values of the redundant reactions determined by neglecting any of the 2 effects.

For this reason, it is not correct to use the reactions obtained for a continuous beam in the design of a similar continuous V. girder. This is also the reason why the proposed method of Edgar Lightfoot goes wrong.

9. The redundant reactions of the continuous V. girder are almost equal to the reactions of the continuous hinged system. This is obviously due to the fact that both systems have almost equal deflections. This is true for concentrated as well as for distributed loads.

10. The redundant reactions of the continuous V. girder can be also obtained by reference to the equivalent beams mentioned under (6).

11. Having found the redundant reactions of the continuous V. girder from the corresponding values of the hinged system, the bending moments and subsequently the normal forces in the different members can be readily found by the panel-method. In this way, the corresponding influence lines can be determined.

This proposed procedure simplifies considerably the design of the continuous Vierendeel girder. It becomes more and more advantageous as the number of spans increases.

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THE ENERGY LOSS IN SOME IRRIGATION WORKS

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INTRODUCTION

It is obvious that the most economical system of irrigation, is that in which no pumping of water is used. It is called the "Free irrigation" system.

In flat countries, such as Egypt, the application of this system requires a great care to keep the water levels, in the water distributing canals, always higher than the cultivated land levels. It is the duty of the hydraulic engineer in general and the irrigation engineer in particular, therefore, to minimise the loss in head along the canal as well as in the distributing work.

The velocity in the canal downstream of the structure is relatively small, and the flow is more or less streaming. The shooting flow leaving the discharge side of the structure meets the slow stream and a "hydraulic jump" is then formed,

It is clear, that, if a hydraulic jump is produced downstream a work, the major part of the energy loss takes place in the jump. For this reason the study will be concentrated in the jump in which the energy is lost and how this loss could be improved.

The most important types of water distributiong works, are the weirs, sluices, pipes and Venturi-flumes.

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The study of the problem, required an accurate measurement of the velocities and pressures in the direction of the stream lines.

The plane between the moving water at the jump, and the rolling water (the roller) for different discharges and different types of Venturi-flumes was determined. The shear stresses at that plane were accurately calculated.

These measurements and calculations were used in determining the different forces acting on the flow at the position of the jump (Fig. 3). The amounts and directions of these forces and their conditions of equilibrium lead to the interesting result. that, if the bed under the jump is given an upward slope in the direction of flow forming a hump, the energy loss in the jump and consequently in the irrigation work itself will be reduced to a great extent.

CHAPTER I

The Energy Loss in the Jump

It is obvious that the loss of energy which is lost partly in the rotation of the roller and partly in the formation of the eddies, is drawn continuously from the energy of the live water. The transfer of the energy from the "live water" to the "eddy dead water" takes place through the mechanism of momentum transfer creating shearing stresses between both layers. In other words the energy loss in the jump depends to a great extent on those created shearing stresses. In order to minimise the energy loss in the jump a trial for reducing the shear stresses will be necessary.

Since the amount of the shearing forces depend on the amount and direction of all forces acting on a vertical strip across the jump (Fig. 3), the evaluation of those forces and the conditions of their equilibrium must be first determined.

1. *The condition of equilibrium of the forces acting on a strip across the jump :*

Applying the momentum theorem, the horizontal component of the forces acting upon the fluid within the control surface enclosing

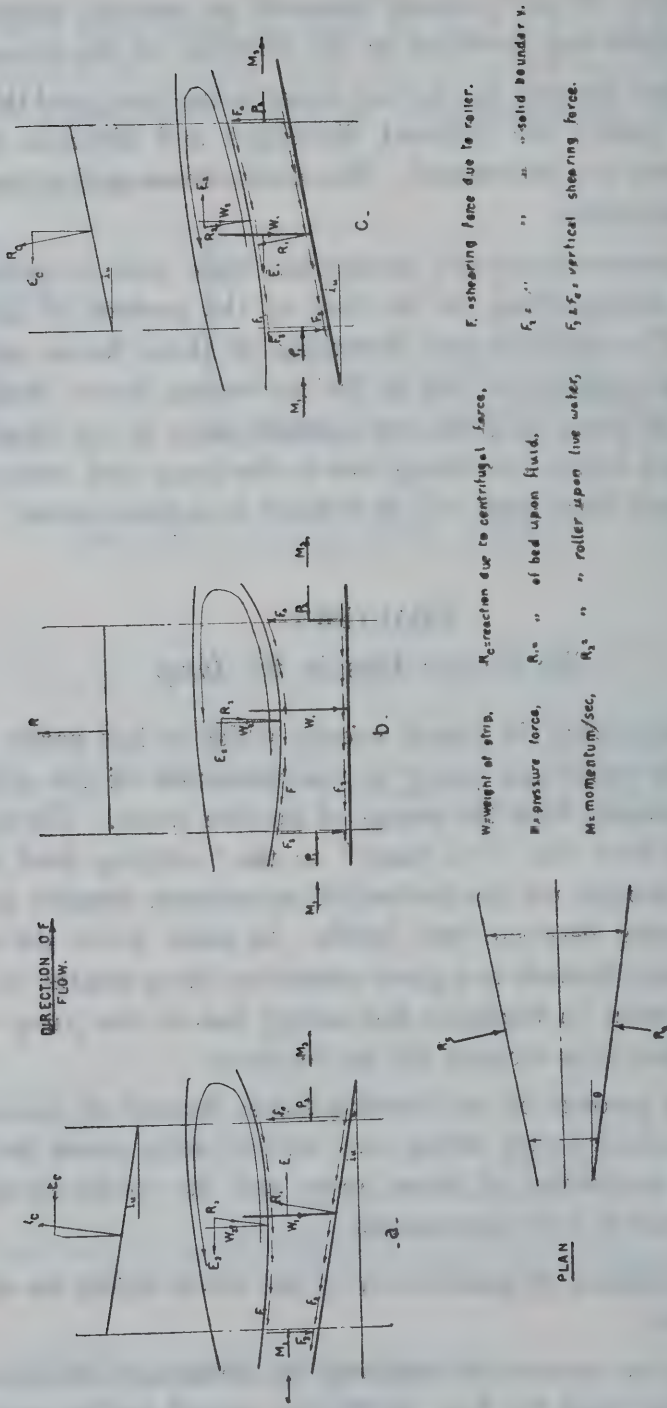


Fig. 3

the live water for any given strip (Fig. 3), will be related by the following equations :

$$\left. \begin{aligned}
 (a) \quad & P_1 - P_2 + E_1 \pm E_2 + E_c - F'_1 - F'_2 + R \\
 & = M_2 - M_1 \quad (\text{Fig. 3 } a) \\
 (b) \quad & P_1 - P_2 \pm E_2 - F'_1 - F'_2 + R \\
 & = M_2 - M_1 \quad (\text{Fig. 3 } b) \\
 (c) \quad & P_1 - P_2 - E_1 \pm E_2 - F'_1 - F'_2 - E_c + R \\
 & = M_2 - M_1 \quad (\text{Fig. 3 } c)
 \end{aligned} \right\} \quad (1)$$

in which

1. The pressure forces P_1 and P_2

Taking $\left(\frac{P'}{\gamma} + z \right)_m$ = the mean pressure head of the live water at any vertical section, s = the height of the "eddying dead water" at the considered section excluding the air bubbles, then the pressure distribution diagram in the live water take the shape shown in (Fig. 4).

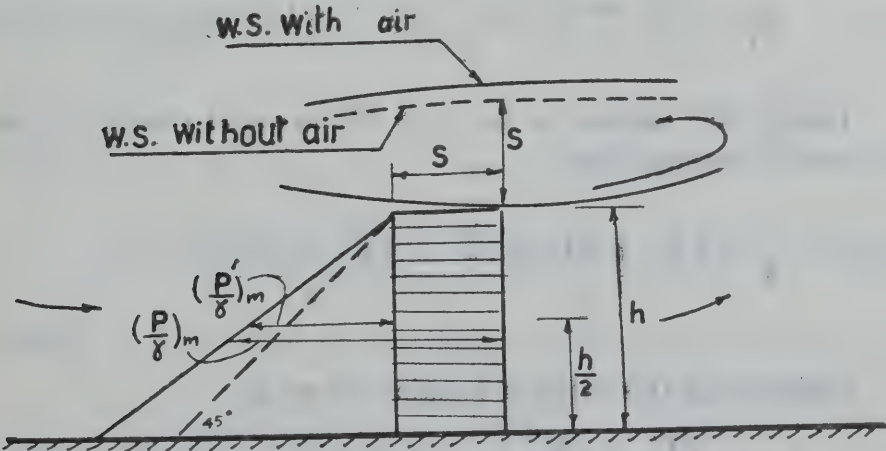


FIG 4.

Then, $P' = \left(\frac{P'}{\gamma} \right)_m \times h + s h$ (2)

assuming that $\left(\frac{P'}{\gamma} \right)_m$ occurs at $\frac{h}{2}$ from the bed, and

putting $\left(\frac{P'}{\gamma} - z \right)_m + s = D$ (3)

then $D = \left(\frac{P'}{\gamma} \right)_m + \frac{h}{2} + s$

or $\left(\frac{P'}{\gamma} \right)_m = D - \frac{h}{2} - s$

therefore $P = (D - \frac{h}{2} - s) h + s h$

$= D h - \frac{h^2}{2} - s h + s h$

or $P = D h - \frac{h^2}{2}$ (4)

2. The horizontal components E_1 and E_2

$$E_1 = \frac{(h_1 + s_1) b_1 + (h_2 + s_2) b_2}{2} \times l \tan i_u \quad (5)$$

Taking " β " defined as the ratio of the actual pressure to the hydrostatic pressure, then,

$$\beta = \frac{\left(\frac{P'}{\gamma} + Z \right)_m \times \frac{h}{2}}{\frac{h^2}{2}} = \frac{\left(\frac{P'}{\gamma} + Z \right)_m}{h}$$

Substituting this value in equation (3) we get,

$$D = \left(\frac{P'}{\gamma} + Z \right)_m + s = \beta h + s \text{ or } s = D - \beta h \quad (6)$$

Substituting this value of “s” in equation (5) for “ E_1 ” we get :

$$E_1 = \frac{\gamma}{2} \left[(h_1 + D_1 - \beta_1 h_1) b_1 + (h_2 + D_2 - \beta_2 h_2) b_2 \right] l \tan i_u = \frac{\gamma}{2} \left\{ \left[D_1 - (\beta_1 - 1) h_1 \right] b_1 + \left[D_2 - (\beta_2 - 1) h_2 \right] b_2 \right\} l \tan i_u \quad (7)$$

and,
$$E_2 = \frac{s_1 b_1 + s_2 b_2}{2} l \frac{\gamma}{2} \tan i_s - \frac{\gamma}{2} \left[(D_1 - \beta_1 h_1) b_1 + (D_2 - \beta_2 h_2) b_2 \right] \tan i_s \quad (8)$$

3. The horizontal components E'_1 and F'_2

The horizontal component of the shearing force (between the live stream and the roller) is :

$$F'_1 = \frac{\tau_s (b_1 - b_2)}{\cos i_s} l \cos i_s = \tau_s l \frac{(b_1 - b_2)}{2}$$

3_b The horizontal component of the friction force at the bed and sides of the channel F_2 is:

$$F_2 = \tau_o A \cos i_u,$$

where τ_o = shear stress at the solid boundary,

A = the area of the live water with the solid boundaries.

Putting $\tau_o = \frac{1}{2} \rho_o v_m^2 f_b \times A \cos i_u \quad (9)$

and using the Chezy's formula $C = \sqrt{\frac{2g}{f_b}}$

where $C = \text{Cheyz's coefficient} = 60$ (in our case).

$$\text{Then } 60 = \sqrt{\frac{2g}{f_b}} \text{ or } f_b = 0.0054.$$

4. The Momenta M_1 and M_2

The momentum $m v_m = \frac{W}{g} v_m$ will be multiplied by a coefficient α' where,

$$\alpha' = \int \frac{v^2 dh}{v_m^3 \cdot h} \quad \text{where,}$$

v_m is the horizontal component of the actual mean velocity

$$V_m \text{ i.e. } V_m = v_m \sec i_m$$

5. The horizontal force E_c

It is a known fact that with curved stream lines the pressure distribution will deviate from the usual hydrostatic one. With concave stream lines as for the case of the live water within the jump, the total pressure along the bed will be higher than its hydrostatic value. This force can be estimated with the help of the pressure distribution diagrams obtained for traverses 1-1 and 2-2. Taking the dynamic pressure at the bed as $\left(\frac{\Delta p}{\gamma}\right)_{c_1}$ and $\left(\frac{\Delta p}{\gamma}\right)_{c_2}$ at traverses 1 and 2 respectively, (Fig. 5), then the average value will be,

$$\left(\frac{\Delta p}{\gamma}\right)_{c_{1,2}} = \frac{1}{2} \left[\left(\frac{\Delta p}{\gamma}\right)_{c_1} - \left(\frac{\Delta p}{\gamma}\right)_{c_2} \right]$$

The total dynamic force acting upon the fluid can be put as :

$$R_c = \gamma \left[\frac{\left(\frac{\Delta p}{\gamma}\right)_{c_1} b_1 + \left(\frac{\Delta p}{\gamma}\right)_{c_2} b_2}{2} \right] \times \frac{1}{\cos i_u}$$

where, l = horizontal distance between traverses 1 and 2
 b_1 and b_2 = breadth of channel.

The horizontal component of this force, E_c which counts for the effect of the curvature and inclination of the stream lines is :

$$E_c = \frac{1}{2} \gamma \left[\left(\frac{\Delta p}{\gamma} \right)_{c_1} b_1 + \left(\frac{\Delta p}{\gamma} \right)_{c_2} b_2 \right] \frac{1}{\cos i_u} \sin i_u \quad (10)$$

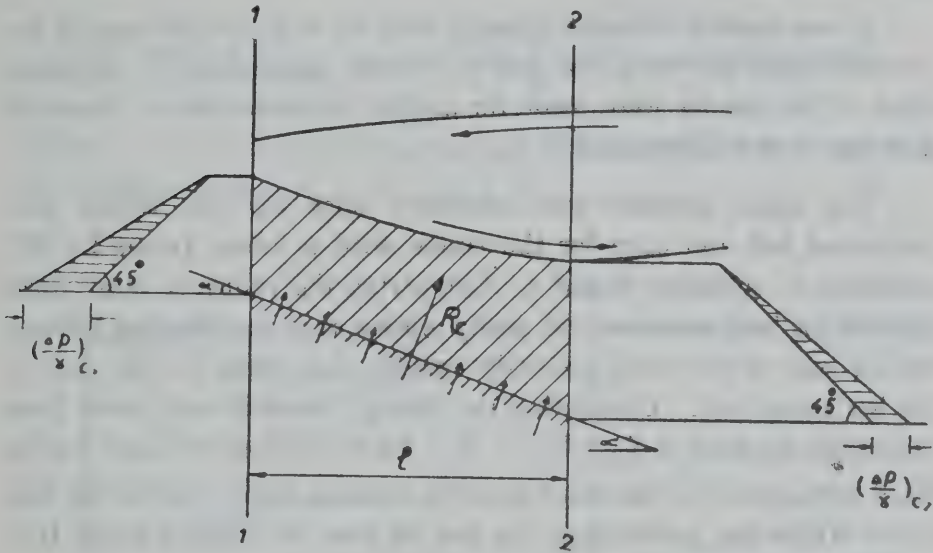


FIG. 5

6. *The horizontal component of the reaction of the two diverging sides upon the water within the control surface considered, (R) :*

$$R = \gamma \left[\left(D_1 h_1 - \frac{h_1^2}{2} \right) + \left(D_2 h_2 - \frac{h_2^2}{2} \right) \right] l \tan \theta \quad (11)$$

The shearing forces at the plane between the live stream and the eddying dead water (the roller) can be obtained from equations (1) as follows :

$$\left. \begin{aligned} (a) \quad F'_1 &= (M_1 + P_1 \pm E_2 + E_o + E_1 + R) - (M_2 + P_2 + F'_2) \\ (b) \quad F'_1 &= (M_1 + P_1 \pm E_2 + O + O + R) - (M_2 + P_2 + F'_2) \\ (c) \quad F'_1 &= (M_1 + P_1 \pm E_2 - E_c - E_1 + R) - (M_2 + P_2 + F'_2) \end{aligned} \right\} \quad (1')$$

Comparing the above values of F_1 obtained from the above three equations, we find that the values obtained from equation "c" which corresponds to the case of the hump has the least value.

This means that, if the bed under the jump is given an upward slope in the direction of flow forming a hump, the loss in HEAD in the jump will be greatly reduced, which is the "key" of this presented work.

It was more convenient to begin with the study of the case of the Venturi-flume in which the flow is "three dimensional". Applications of the results were made for another irrigation work where the flow was "two-dimensional"

The shear stresses were-calculated once for the flume with horizontal bed and once for the flume with a hump (slope 1 : 10), according to equations b and c. Comparing the values of these shear stresses for both cases one can easily notice, that the shearing stresses for the case of the hump are really smaller than those for the case of the horizontal bed. Calculating the Energy losses in both cases from the simple equation $\Delta Z_e = H_1 - H_2$, we found that the small loss in Head corresponds to the small shearing stresses, which realises the idea of the writer for minimising the loss in head by using a hump D.S. the irrigation work.

As the shearing forces are affected by the hump, the path of the energy line must consequently be affected. It will be interesting here, to explain this effect.

2. Effect of the hump on the path of the energy line:

The energy line is that passing through the points of the total energy head in successive sections along the flume. Formulating this definition mathematically for two successive sections (Fig. 6), we get :

The total energy head at section 1,

$$H_1 = \left(\frac{P'}{\gamma} + Z \right)_{m1} + S_1 + \alpha_1 \frac{v_{m1}^2}{2g} \sec^2 i_{m1} \text{ (for horizontal bed).}$$

$$H_1 = \left(\frac{P'}{y} + Z \right)_{m1} + s_1 + y_1 + \alpha_1 \frac{v_{m1}^2}{2g} \sec^2 i_{m1} \text{ (for the hump).}$$

and the total energy head at section II:

$$H_2 = \left(\frac{P'}{y} + Z \right)_{m2} + s_2 + \alpha_2 \frac{v_{m2}^2}{2g} \sec^2 i_{m2} \text{ (for horizontal bed).}$$

$$H_2 = \left(\frac{P'}{y} + Z \right)_{m2} + s_2 + y_2 + \alpha_2 \frac{v_{m2}^2}{2g} \sec^2 i_{m2} \text{ (for the hump).}$$

where: y = the height of the sloping bed above the horizontal bed at the considered section.

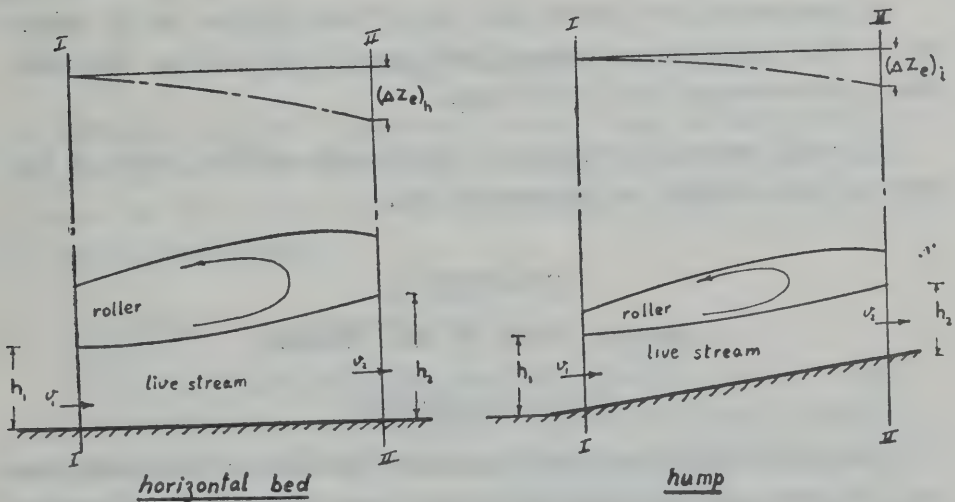


FIG. 6

Comparing each pair of the energy lines (horizontal bed and hump), one can notice, that the energy line corresponding to the flume with hump is flatter than that with horizontal bed.

i.e., $\frac{(\Delta Z_e)_i}{l}$ is less than $\frac{(\Delta Z_e)_h}{l}$

where: $(\Delta Z_e)_i$ is the loss in head along a strip l for the flume with hump.

and $(\Delta Z_e)_h$ is the loss in head along the same strip for the flume with horizontal bed.

It is also clear from the different figures that the total loss in head, for the case of hump, $(\Sigma \Delta Z_e)_i$ is less than that for the case of horizontal floor, $(\Sigma \Delta Z_e)_h$.

Going through the details, we find that the path of the energy line in the Venturi-flume and consequently the total energy loss in the jump is affected by many factors namely, the throat ratio, the degree of divergence of the downstream sides, and the upward bed slope under the jump.

Experiments have shown that the 1st two factors are of minor order while the third one namely, the slope of the bed has the main effect on the energy loss in the jump.

For this reason the writer will concentrate on the study of the effect of the bed slope on the energy loss.

CHAPTER II

Applied Study of the Problem

§ 1.—INTRODUCTION

From the above theoretical treatment, we come to the result, that by giving the bed below the jump an upward slope in the direction of flow, the energy loss will be reduced. We want here to show how this reduction is physically produced.

A hydraulic jump is, so to speak, a transition phenomenon between two distinct types of flow *i.e.*, shooting and streaming, through which the flow changes gradually from its low level to a high level, and can never be expected to be sudden, otherwise the jump would form a vertical rise in the water surface, which is absurd. The low and high levels upstream and downstream of the jump

necessitates that water will flow under gravity backwards towards the shooting stream forming what is usually referred to as the roller or dead water.

It was explained before that the shear stresses, between live and dead water, which are responsible of the head loss in the jump are directly dependant on the weight of the roller itself. If it could be possible to reduce the weight of the roller the shear stresses and consequently the loss in head would certainly be reduced. This could be reached, if the live stream is gradually elevated in such a way that only a smaller quantity of water is allowed to fall backwards on its surface forming a small roller, and hence the weight of the dead water will be less (Fig 7). The produced shear stresses and consequently the energy loss will be smaller than that for the flume with horizontal bed.

§ 2.—TYPES OF FLOW CAUSED BY THE HUMP

Remark: “(In all experiments carried out in this work, the downstream was raised up to a level allowing the flow to be the limit between the free and drowned conditions, in order to get the maximum submergence).

(a) *Description of the types of flow:*

In Venturi-flumes with horizontal bed, one jump only is produced in the downstream for each discharge. In Venturi-flumes with hump, however, the cases will be different. Sometimes one jump is produced and sometimes 2 jumps are possible, depending on the commanding head in the upstream.

For example we consider the case of a flume with a throat ratio = 0.5 and divergence = 1.6 and a slope of hump = 1 : 10. For a discharge = 30.0 lit./sec., giving a water depth in the upstream = 22.34 cm. only one jump is produced. Reducing the discharge and consequently the upstream depth, the jump will be divided into two jumps meeting at a point. For further reduction the two jumps separate, one belongs to the Venturi-flume while the other takes place

at the end of the hump, which acts as a weir. Reducing the upstream depth more and more the jump belonging to the flume disappears while that downstream the hump remains acting.

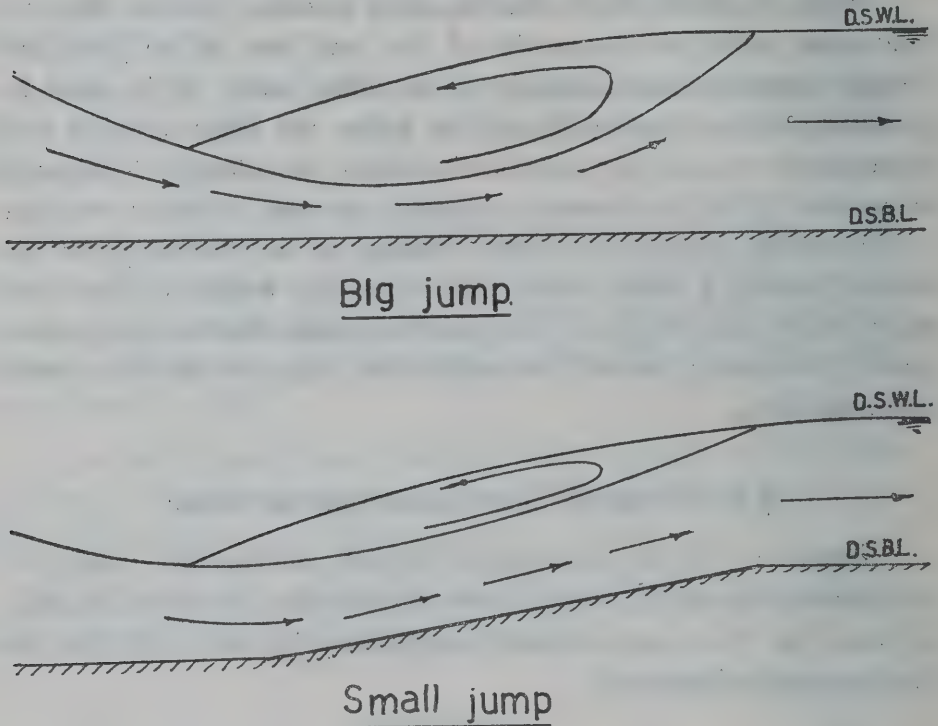


FIG. 7

It has been found that for each slope of hump $\left(\frac{y}{l}\right)$ (Fig. 2) there are two limiting discharges;

1. *The upper limit (a):*

Is that which produces one jump on the hump. If this discharge is increased, we obtain still one jump. On the other hand if it is decreased the jump begins to separate forming two jumps (Fig. 8)

2. *The lower limit (b):*

For further reduction of the discharge the jump in the downstream of the hump remains in place, while that over the hump diminishes

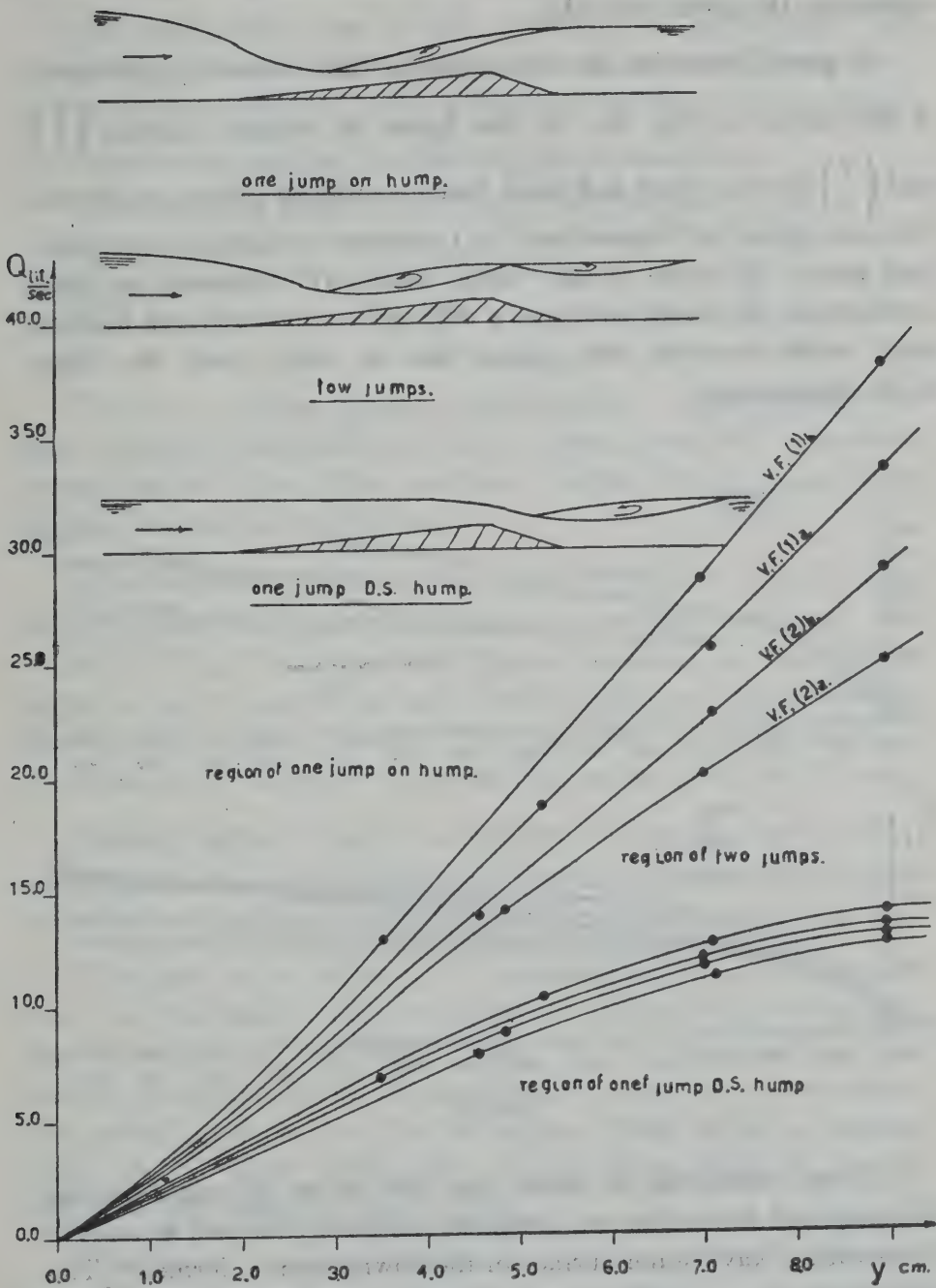


FIG. 8

gradually till it disappears. At that moment the passing discharge represents the lower limit (b).

A good illustration for these types of flow caused by the hump is that shown in (Fig. 9). In this figure the relation between $\left(\frac{y}{h}\right)$ and $\left(\frac{y}{l}\right)$ for the upper and lower limits mentioned before, are plotted. The area below the "upper limit" (a) concerns one jump on the hump, that above the curve of the "lower limit (b)" concerns one jump downstream the hump, acting as a weir and the enclosed area between both curves concerns two jumps one on hump and the other in its downstream.

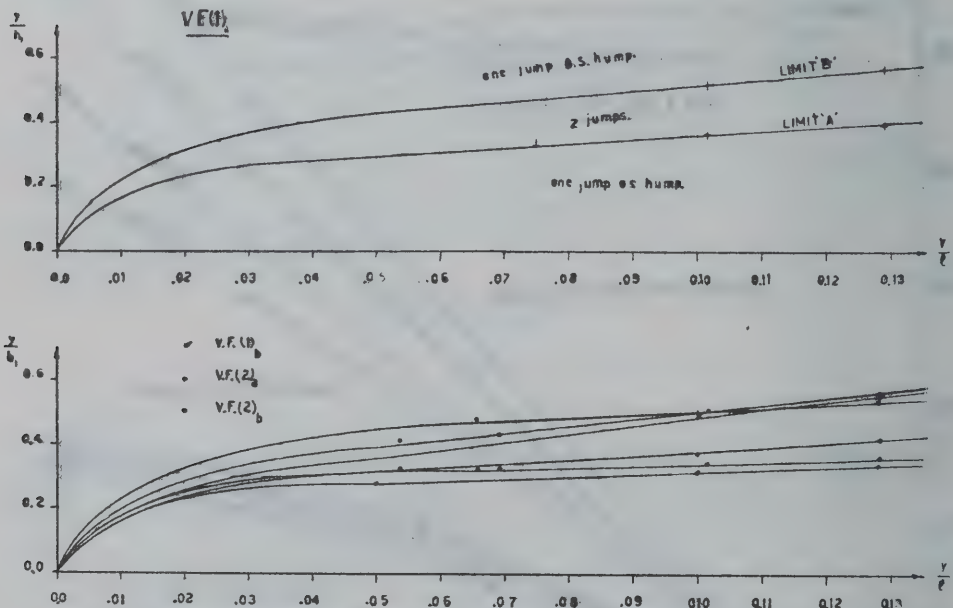


FIG. 9

It was interesting to notice that just below the upper limiting discharge and before the two jumps are completely formed, an undular jump with a wavy water surface in the downstream is formed. This state appears as a transition region between the case of one jump and that of two jumps.

(b) *Behavior of the total energy loss for each type of flow:*

The total energy loss for each discharge passing through each flume was calculated, (the difference between the total energy head in the upstream and that in the downstream of the whole structure).

The flumes used had the following characteristics:—

1. Flume with hroat ratio 0.5 and side divergence 1 : 6
2. " " " " 0.5 " " " 1 : 10
3. " " " " 0.4 " " " 1 : 6
4. " " " " 0.4 " " " 1 : 10

The bed of these flumes was horizontal. These flumes were used again after three defferent humps were suecessively introduced to each one of them. The maximum height of the humps were ranging between 3.5 and 9.0 cms.

The calculated values of the total energy loss ($\Sigma \Delta Z_0$) were plotted against the measured discharge Q , for each flume and were plotted in (Ftg. 10 to 12). Each figure represents a set of curves corresponding to one of the used flumes. Each set consists of four curves. One of them belongs to the horizontal bed while the three others belong to the three humps ranging between (3.5 and 9.0 cms.)

In each figure, it can be noticed that the energy loss ($\Sigma \Delta Z_0$) for the gradually raised bed is always less than that for the horizontal bed whatever the slope is and whatever the type of flow may be.

The part of the curve "d-a" represents the flow forming one jump over the hump, "o-b" represents the flow giving one jump in the downstream of the hump; while "b-c", represents the flow forming two jumps, one on the hump and the other in its downstream. The part of the curve "c-a", is the region within which an undular jump with a wavy downstream surface takes place. It can be considered as a transition state of flow.

The points *a* and *b* indicate the position of the upper and lower limits, respectively (§ 1). Going back to the sets of curves shown in (Figs. 10 to 12), it can be noticed, that the total energy

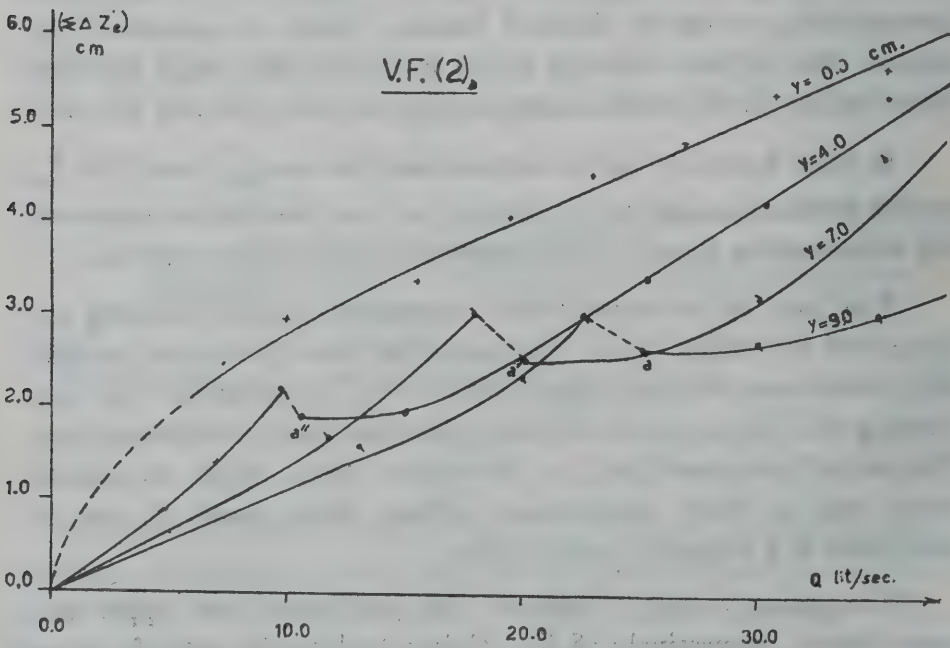
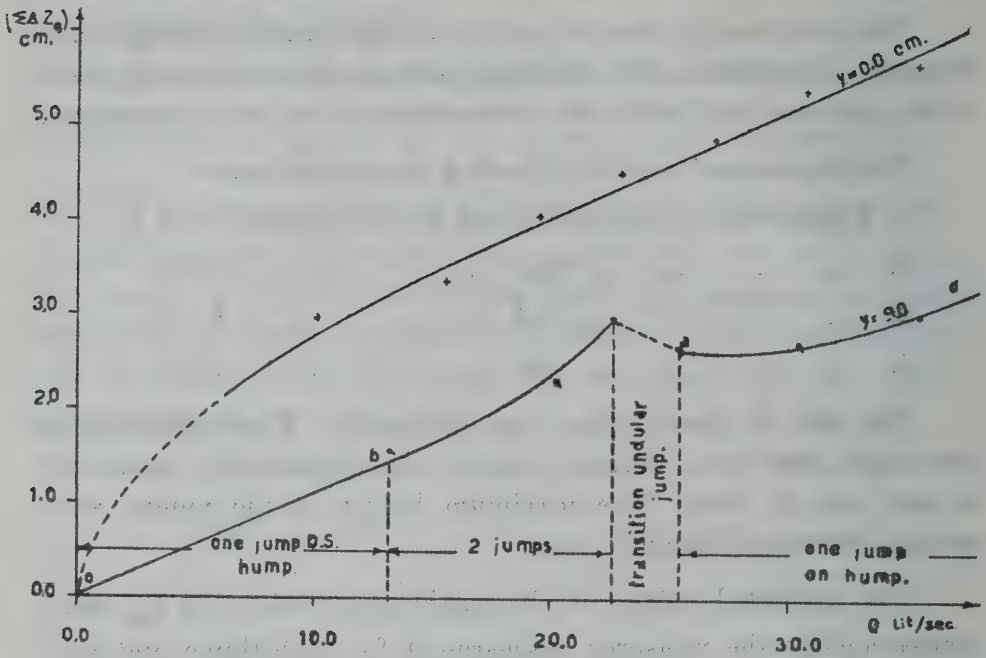
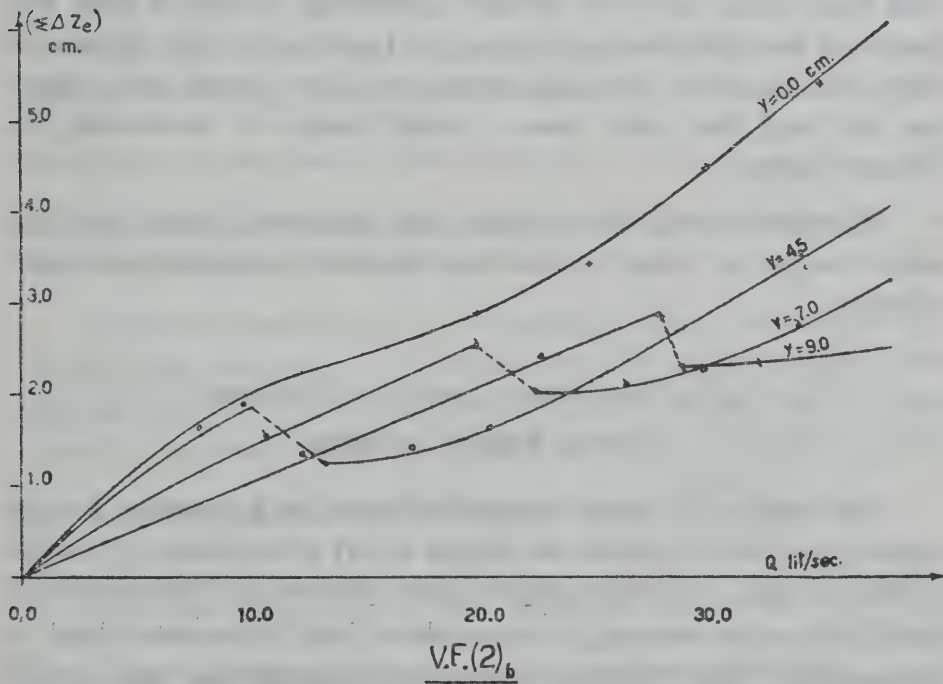
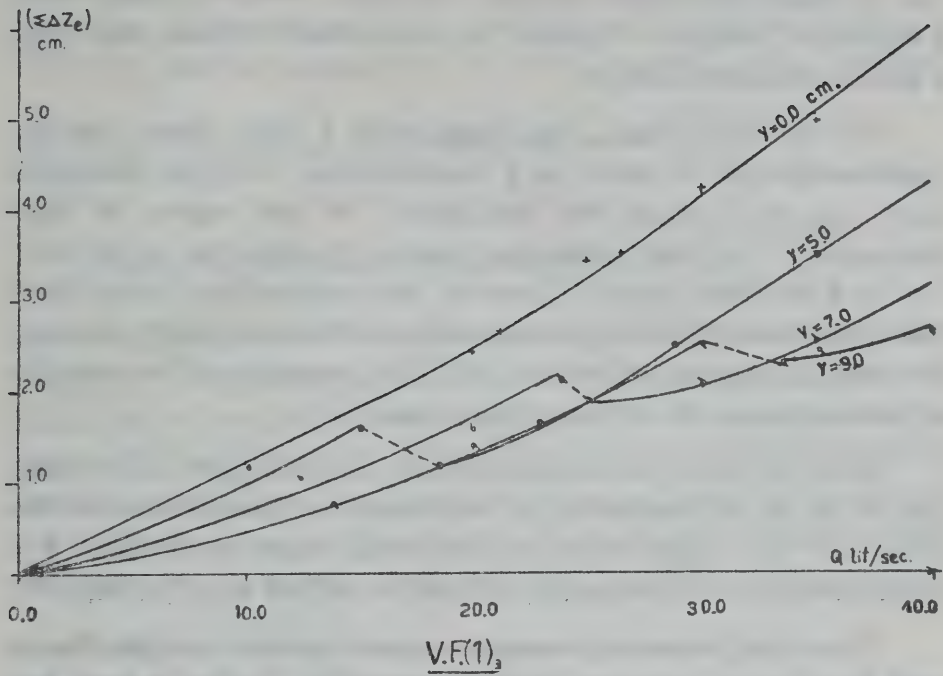


FIG. 10 and 13



FIGS. 11 and 12

loss is small for higher humps. In other words a hump with a maximum height = 9.0 cms. is more effective than that with a smaller maximum height.

On the other hand, the flume with a high hump has the disadvantage that it works as a Venturi-flume for high discharges only, *i.e.*, its function will be limited for the region of high discharges. For low discharges, however, between the upper limit "*a*" and the lower limit "*b*", one of the produced two jumps takes place downstream the hump and causes scour to the bed of the channel. For small discharges, below limit "*b*", the structure will no more be a Venture-flume. It will act just as a weir.

If the height of the hump is reduced we get bigger loss in head, but the region of functioning of the flume as a Venturiflume will be wider and the upper limit will be shifted from (*a*) to (*a'*). For further reduction of the height, (*a'*) will be shifted to (*a''*) (Fig. 13).

From this discussion, one comes to the conclusion that the flumes with high humps are more effective concerning the loss in head, but they have the disadvantage of being in function for high discharges only. On the other hand, the flumes with low humps cause bigger loss in head but they have a wider range of functioning as Venturi-flumes.

Therefore, it will be interesting and important, here to find out which height of hump is therefore the most convenient and more effective.

§ 3.—THE ENERGY LOSS AS A FUNCTION OF THE HEIGHT OF HUMP

The height of hump cannot be taken as a measure for its convenience, simply because the height of any given bump (*y*) can be considered high, if the water depth in the upstream h_1 is comparatively small, and on the contrary, it is considered low, if the water depth is comparatively big. For this reason the dimensionless ratio (y/h_1) will be the right factor used for the comparison between the effect of

the different humps on the energy loss instead of using the height "y". Similarly, the total energy for each hump (by a given discharge) $(\Sigma \Delta Z_e)_i$ will be compared with that of the flume with horizontal bed $(\Sigma \Delta Z_e)_h$ through the dimensionless ratio $\frac{(\Sigma \Delta Z_e)_i}{(\Sigma \Delta Z_e)_h}$.

Plotting the different values of the efficiency $\frac{(\Sigma \Delta Z_e)_i}{(\Sigma \Delta Z_e)_h}$ against the ratio (y/h_1) for different discharges, we get the curves shown in (Fig. 14). Each one of them belongs to one of the used four Venturi-flumes and represents the average values for the three humps ranging between (3.5 and 9.0 cms.). These curves show clearly that by raising the bed of the flume under the jump forming a hump, the energy loss is very much reduced. The reduction ranges between 48 and 60 % of the loss in the flume with horizontal bed. In other words the minimum value of $\frac{(\Sigma \Delta Z_e)_i}{(\Sigma \Delta Z_e)_h}$ varies between 0.4 and 0.52.

Comparing the curves with each other we find, that the flume with the throat ratio 0.5 and side divergence 1 : 10 is the most efficient one because it possesses the least minimum value of $\frac{(\Sigma \Delta Z_e)_i}{(\Sigma \Delta Z_e)_h}$. This is to be expected because the energy loss in general is less for flumes with flatter side divergence.

Comparing now curve No.1 belonging to flume with throat ratio 0.5 and side divergence 1 : 10 with curve No. 2 belonging to flume with throat ratio 0.4 and side divergence 1 : 10 we find that the latter is more convenient because it is flatter within a comparatively large zone about the position of the minimum ratio $\frac{(\Sigma \Delta Z_e)_i}{(\Sigma \Delta Z_e)_h}$.

Since the difference in the minimum values of $\frac{(\Sigma \Delta Z_e)_i}{(\Sigma \Delta Z_e)_h}$ (of curves 1 and 2) is small, we come to the result that flume 2 with throat ratio 0.4 will be the most convenient one. The minimum

values of $\frac{(\sum \Delta Z_e)_i}{(\sum \Delta Z_e)_h}$ for the different flumes corresponds to values of $\frac{y}{h_1}$ between 0.26 and 0.38 (Fig. 14).

4. Design of the most efficient flume:

Figure 14, shows that the efficient flume is that which produces the least value of $\frac{(\sum \Delta Z_e)_i}{(\sum \Delta Z_e)_h}$. In that figure the effect of the throat ratio and that of the side divergence were neglected because their effect was of minor order compared with that of the hump.

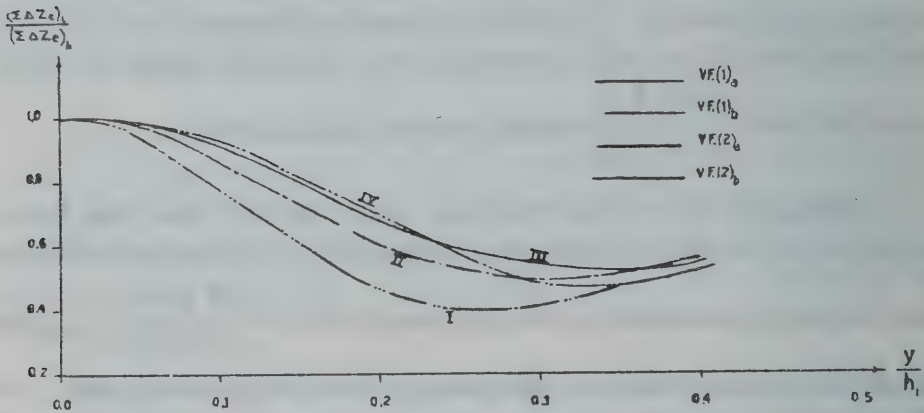


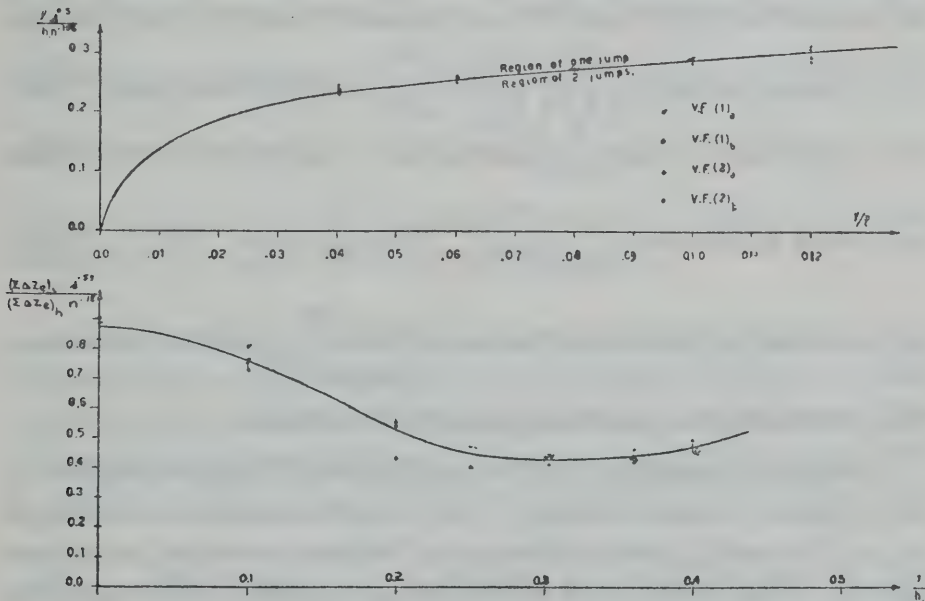
FIG. 14

Since we intend, here, to design the most efficient flume, the effect of the throat ratio and that of the side divergence on the energy loss had to be introduced. For this reason, the ratio (y/h_1) was improved by introducing these effects (by elimination). The modified form of (y/h_1) became $(y \cdot S^{0.5}/h_1 \cdot n^{0.106})$; and that of the ratio $\frac{(\sum \Delta Z_e)_i}{(\sum \Delta Z_e)_h}$

$$\text{became } \frac{(\sum \Delta Z_e)_i}{(\sum \Delta Z_e)_h} \times \frac{S^{0.33}}{n^{0.18}}.$$

The modified curves which are going to be used in the design of the most efficient flume were drawn in (Figs. 15 and 16) instead of those shown in (Figs. 9 and 14). The curve $(ys^{0.5}/h, n^{0.106})$ against (y/l) represents, as explained before, the limit between the region of one jump on hump and that of the two jumps.

If the maximum gain of energy is required, the design must be carried out for the region of one jump on hump. That is because the rate of change of the energy loss $(\Sigma \Delta Z_e)$ with respect to Q , for the region of one jump, is less than that for the region of 2 jumps, as represented by the slopes of curve "a-d" and curve "c-b" respectively (Fig. 13).



FIGS. 15 and 16

For the region of one jump on hump the relation between $\frac{(\Sigma \Delta Z_e)_i}{(\Sigma \Delta Z_e)_h} \times \frac{S^{0.55}}{n^{0.18}}$ and (y/h_1) was given in (Fig. 16). Fortunately the

S = the throat ratio.

n = the side divergence.

zone about $\frac{(\sum \Delta Z_e)_i}{(\sum \Delta Z_e)_h} \times \frac{S^{0.55}}{n^{0.18}} = \text{min.}$ is flat, so that any value of (y/h_1) within that zone gives more or less the same effect on the energy loss.

Procedure of design:

Generally, the form of the flume (especially the throat ratio "S" and side divergence "n"), as well as the discharge, will be given. In this case the corresponding water depth in the upstream h_1 can be approximately determined for the preliminary design. Required will be the dimensions of the hump which gives the least loss in head for the region of one jump on hump i.e., "y" and "l". For this purpose the two curves (Figs. 15 and 16) will be used. From (Fig. 16), the ratio

(y/h_1) corresponding to $\frac{(\sum \Delta Z_e)_i}{(\sum \Delta Z_e)_h} \times \frac{S^{0.55}}{n^{0.18}} = 0.3$ will be chosen. Since

" h_1 " is given, $y = 0.3 h_1$ is determined. The values of "y" is then

introduced in the ratio $\left(\frac{y}{h_1} \frac{S^{0.5}}{n^{0.106}} \right)$. Making use of (Fig. 15), (y/l)

corresponding to the obtained values of $\left(\frac{y}{h_1} \frac{S^{0.5}}{n^{0.106}} \right)$ will be determined.

from which, the length of the hump "l" is obtained.

Practically speaking, the Venturi-flume will not be designed to pass a certain constant discharge. On the contrary different discharges ranging between a given minimum and maximum values have to pass through. The design must therefore be convenient for the whole range and not limited for a certain discharge.

Now, a question arises, : which discharge will be chosen as a basis for the design? In other words which upstream depth " h_1 " is to be introduced in the ratio (y/h_1) in order to give the required height of hump "y" and length?

If the maximum " h_1 " corresponding to max. "Q", is introduced, therefore $y = 0.3 (h_1)_{\text{max.}}$

and
$$\frac{y s^{0.5}}{h_1 n^{0.106}} = \frac{0.3 h_{l_{\max.}} s^{0.5}}{h_{l_{\max.}} n^{0.106}} = \frac{0.3 s^{0.5}}{n^{0.106}}$$

and the corresponding (y/l) from (Fig. 15), is obtained, from which "l" is determined. Thus the dimensions of the hump "y" and "l" for the maximum discharge are fixed up. For example for $s = 0.4$ and

$n = 0.1$, (y/h_1) being equal to 0.3 (Fig. 16); $\left(\frac{y s^{0.5}}{h_1 n^{0.106}}\right) = \frac{0.3 \times 0.632}{0.783} = 0.243$, From curve (Fig. 15); the corresponding $(y/l) = 0.05$.

If now a smaller discharge than " $Q_{\max.}$ " passes, the corresponding " h_1 " will also be smaller than $(h_1)_{\max.}$, with the result that $\left(\frac{y s^{0.5}}{h_1 n^{0.106}}\right)$ will be bigger. But since the ratio (y/l) is fixed by 0.05, the point having the ordinates $(y/l) = 0.05$ and $\left(\frac{y s^{0.5}}{h_1 n^{0.106}}\right) > 0.243$ lies above the curve and falling in the region of the two jumps, which was excluded for the most efficient hump. (Fig. 15).

This shows that we should not design for the maximum discharge, otherwise the so-designed hump will be efficient for the maximum discharge only. For this reason we are going to try the use of the minimum discharge as basis. In this cases the discharge passing will be more than " $Q_{\min.}$ " and the corresponding h_1 will be more than $(h_1)_{\min.}$. The ratio $\left(\frac{y s^{0.5}}{h_1 n^{0.106}}\right)$ will be less than 0.243 and the point giving $(y/l) = 0.05$ lies below the limiting curve and falls in the region of the one jump, which is recommended.

Although the ratio (y/l) remains the same for all discharges yet the ratio (y/h_1) varies with the upstream depth " h_1 ". for discharges more than " $Q_{\min.}$ ", (y/h_1) decreases and moves on the curve (Fig. 16) towards the origin 0, beginning from $(y/h_1) = 0.3$. The rate of charge

of $\frac{(\sum \Delta Z_e)_i}{(\sum \Delta Z_e)_h} \propto \frac{s^{0.55}}{n^{0.18}}$ with respect to (y/h_1) begins to be of a major order after $y/h_1 = 0.2$, giving a small range for (y/h_1) to oscillate between 0.5 and 0.2 with the result that for the big discharge (y/h_1) will be less than 0.2 giving a big value of $\frac{(\sum \Delta Z_e)_i}{(\sum \Delta Z_e)_h} \times \frac{s^{0.55}}{n^{0.18}}$ and consequently a big loss in head.

In order to let big discharges to pass with small energy losses the range within which (y/h_1) oscillates must be widened. This is reached to by shifting the chosen value of (y/h_1) from 0.5 to the upper limit of the flat portion of the curve which seems to be equal to 0.4 approximately. The range in this case will be between 0.4 and 0.2 instead of being between 0.3 and 0.2, allowing the hump to be efficient for a wide range of discharges.

Solved Example:

A Venturi-flume of the type shown in (Fig. 17), is constructed along a rectangular channel, the maximum discharge of which is $5.55 \text{ m}^3/\text{sec}$. and minimum discharge is $2.0 \text{ m}^3/\text{sec}$. Its breadth = 3.0 ms., and throat breadth = 1.20 ms. Side walls divergence 1:8.

If the corresponding upstream depths " h_1 " are 1.05 and 1.13 ms. respectively, it is required to design the most efficient hump. For the most efficient hump the ratio $y/(h_1)_{\min.} = 0.4$ (Fig. 16).

$$\text{i.e. } y/1.13 = 0.4 \quad \therefore y = 0.45 \text{ ms.}$$

$$\text{and as } s = 0.4 \quad \therefore s^{0.55} = 0.633$$

$$n = 0.125 \quad n^{0.106} = 0.802$$

To determine the length of hump (1) we substitute in the equation

$$\frac{y s^{0.5}}{h_1 n^{0.106}} = 0.2 - 0.093 (y/l) \text{ (Fig. 15).}$$

$$0.316 = 0.2 - 0.093 (y/l)$$

from which $y/l = 0.125$ $\therefore l = 3.60 \text{ ms.}$

From the maximum discharge,

$$y/h_1 = \frac{0.45}{1.85} = 0.243$$

which lies within the convenient range mentioned before.

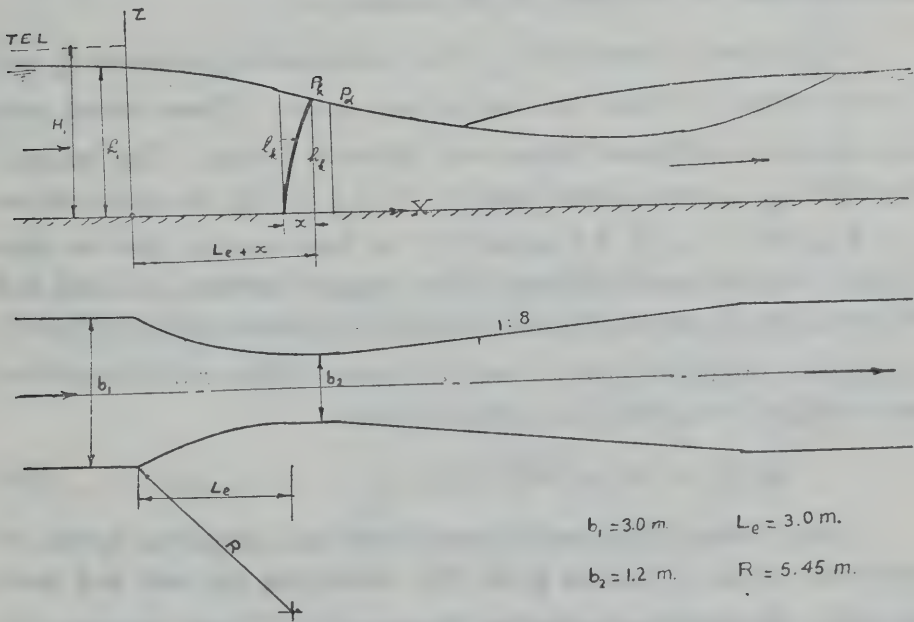


FIG. 17

Hence the hump is efficient for different discharges ranging between the values of $Q_{\min.}$ and $Q_{\max.}$ corresponding to $y/h_1 = 0.2$ and 0.4 respectively.

CHAPTER III

Measurement of any Discharge Passing Through the Venturi-Flume with Hump

1.—EQUATION OF DISCHARGE

After the most efficient Venturi-flume is designed and constructed, it will be important to know how any discharge passing through the flume is measured.

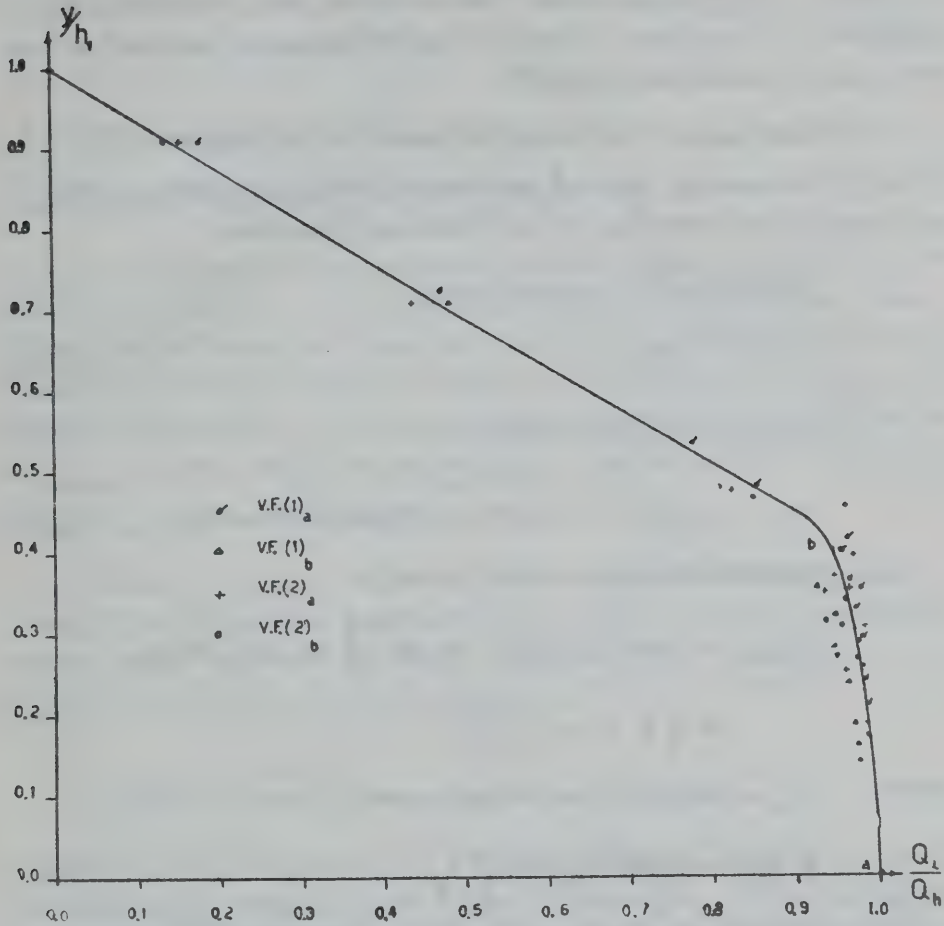
The known equations for computing the discharge through flumes with horizontal bed cannot be used here, without introducing the effect of the hump. For a given upstream water depth h_1 , if the corresponding discharge through the flume with horizontal bed $= Q_h$ and that through the flume with hump $= Q_i$, then the ratio Q_i / Q_h represents the effect of the hump on the discharge.

For different values of " h_1 " the corresponding values of Q_h and Q_i were obtained from the calibration curves. These curves were constructed for different flumes and different humps. The value of the ratio Q_i / Q_h were plotted against (y / h_1) giving the curve shown in (Fig. 18). In (§ 4 Chapter II) we have shown that the ratio (y / h_1) for the most efficient flume ranges between 0.2 and 0.4. Fortunately, all points of the curve (Q_i / Q_h) against (y / h_1) (Fig. 18), within this range fall about a very flat curve which can be considered as a straight line giving the simple equation:

$$Q_i / Q_h = 1.0 - 0.20 (y / h_1). \quad \dots \dots \dots (14)$$

If the Venturi-flume with hump is used as a measuring device, the upstream depth h_1 will be given (by measuring the bed and water levels). In order to calculate the discharge " Q_i " (from equation 14); " Q_h " for the same upstream depth " h_1 " must first be determined.

The determination of " Q_h " follows according to the equation obtained by A. Khafagi (7), in his work "Der Venturi-kanal (Theorie und Anwendung)". The advantage of this equation is that the effect



FOR PART "a-b"

$$\frac{Q_1}{Q_h} = 1.0 - 0.2 \frac{y}{h}$$

FIG. 18

of the shape of the entrance, exit walls as well as throat ratio was taken into consideration. He proved in the theoretical treatment leading to that equation that, in case of curved streamline flow, the critical depth is not a vertical depth, but a curve which passes by all points where the velocity is equal to the critical velocity. This curve is a curve with equal $(p/\gamma + Z)$ and perpendicular to the streamlines.

It describes (for the case of the Venturi-flume with horizontal bed) a parabola. It was found that this curve intersects the bed at the beginning of the throat (Fig. 17).

On these basis, if the depth of water in the upstream " h_1 " is given (by measuring the bed and water levels), the discharge " Q_h " can be calculated according to the following procedure :

$$1. 0.148 s^2 t^3 - t + 1 = 0 \quad . \quad . \quad . \quad (15)$$

$$2. H = t h_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$3. h_k = 0.73H = 0.73 t h_1. \quad . \quad . \quad . \quad . \quad (17)$$

$$4. a = \sqrt{L_e^2 + (2.93 t - 2.14 t^2) h_1^2} - L_e \quad . \quad (18)$$

(a = parameter of the critical curve)

$$5. l_k = 1/4a \left\{ 2a h_k \sqrt{1 + 4a^2 h_k^2} + \log_e [2a h_k + \sqrt{1 + 4a^2 h_k^2}] \right\} \quad . \quad . \quad . \quad . \quad (19)$$

where (l_k = length of the critical curve)

$$6. Q_h = 2.295 b_2 \sqrt{t h_1} \cdot l_k \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

All notations are shown in (Fig. 17). Equation (20) giving the discharge passing through the flume with horizontal bed for the given upstream depth h_1 was derived as follows :

$$Q_h = v_k \cdot l_k \cdot b_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

v_k = critical velocity, l_k = length of critical curve and

b_2 = width of throat.

$$\text{On the other hand : } \frac{v_k^2}{2g} = H - h_k = H - 0.73H = H (1-0.73)$$

therefore ; $v_k^2 / 2g = 0.27 H$.

but; $H = th_1$ therefore; $v_k^2/2g = 0.27 th_1$

$$\text{or } v_k = \sqrt{2g} \times \sqrt{0.27 th_1} \quad (22)$$

$$Q_h = \sqrt{2g} \sqrt{0.27 th_1} l_k \cdot b_2$$

therefore $Q_h = 2.295 b_2 \sqrt{th_1} \times l_k$ (Equ. 20).

The advantage of this equation is that it contains the effect of the shape of the flume as well as the effect of the curvature of the stream-lines. In the same time it does not contain any coefficient of discharge which is generally a source of trouble for the designer, because it is variable. Equation (20) gives therefore the calculated discharge $Q_{\text{theoretical}} = \text{the actual discharge } Q_{\text{act.}}$ i.e. the coefficient of discharge $Q_{\text{theo.}}/Q_{\text{act.}}$ equals always unity.

Coming back to equation (14); the required discharge passing through the flume with hump, corresponding to the given upstream water depth " h_1 " can be easily determined after substituting the values of " Q_h " from equation (20).

2. Solved Example 2:

In the solved example (1), the most efficient hump was designed for a given flume and a given range of discharges. Here, assuming that this designed flume is constructed for the purpose of measuring the discharge, it will be required to calculate the discharge Q_i , passing at any time, corresponding to any measured upstream water depth " h_1 ". Using now the same flume designed in example (1) and given in (Fig. 17), calculate the discharge Q_i for a given $h_1 = 1.6$.

Steps of the calculation:

Using the equation $Q_i/Q_h = 1.0 - 0.2 (y/h_1) \dots$ (Equ. 14).
where $Q_i = \text{discharge of the flume with hump corresponding to upstream depth } "h_1"$.

Q_i = discharge of the horizontal flume corresponding to the same upstream depth " h_1 ".

In the above equation both y and h_1 are known ($y = 0.45$ and $h_1 = 1.60$ ms.). To determine " Q_i ", it is required therefore to calculate " Q_h ".

Calculation of " Q_h ".

Referring to equations (15, 16, 17, 18, 19 and 20) (Fig. 70), we get

$$1. 0.148 s^2 t^3 - t + 1 = 0$$

$$\text{as } s = 0.4 \quad \therefore s^2 = 0.16$$

$$\text{therefore: } 0.148 \times 0.16 t^3 - t + 1 = 0 \quad \therefore t = 1.025$$

$$2. H = th_1 = 1.025 \times 1.60 = 1.64 \text{ ms.}$$

$$3. h_k = 0.73 H = 0.73 \times 1.64 = 1.195 \text{ ms.}$$

$$4. a = \frac{\sqrt{L_c^2 - (2.95t - 2.14t^2)h_1} - L_c}{1.07t^2h_1^2}$$

$$= \frac{\sqrt{9.0 + (0.75) \times 2.55 - 3.0} - 0.3}{1.07 \times 1.05 \times 2.55} = \frac{0.3}{2.87} = 0.1045.$$

$$5. l_k = 1/4a [2a h_k \sqrt{1 + 4a^2 h_k^2} + \log_e (2a h_k + \sqrt{1 + 4a^2 h_k^2})]$$

$$= 1/4 \times 0.1045 [2 \times 0.1045 \times 1.195 \sqrt{1 + 4 \times (0.1045)^2 \times (1.195)^2}$$

$$+ \log_e (2 \times 0.1045 \times 1.195 + \sqrt{1 + 4 \times (0.1045)^2})]$$

$$= 1/0.418 [0.258 + 0.247] = 0.505/0.418 = 1.21 \text{ ms.}$$

$$\begin{aligned}
 6. \quad Q_h &= 2.295 \, b_2 \sqrt{t \, h_1} \cdot l_k \\
 &= 2.295 \times 1.2 \sqrt{1.64 \times 1.21} = 4.27 \, \text{m}^3/\text{sec.}
 \end{aligned}$$

Now using equation (14);

$$Q_i/Q_h = 1.0 - 0.2 (y/h_1)$$

$$Q_i/4.27 = 1.0 - 0.2 (0.45/1.60). \quad \underline{Q_i = 3.98 \, \text{m}^3/\text{sec.}}$$

The same procedure can be followed for calculating different values of “ Q_i ” for different upstream water depths “ h_1 ”. The calculated values can be plotted against each other giving a curve which can be used with advantage as a calibration curve for the designed flume.

CONCLUSION

The theoretical and practical treatment of the problem showed that the energy loss in the Venturi-flume could be reduced to about 54%, if the bed under the jump is given an upward slope in the direction of flow forming a hump. This reduction in the energy loss (about 55%) which can be considered as a gain in the energy head (compared with the case of flume with horizontal bed), corresponds to a ratio of about 0.30 between the height of the hump “ y ” and the maximum water depth in the upstream “ h_1 ”, (Fig. 1).

A similar gain in the energy head could be obtained in the case of the sluice gate (Fig. 2), when the hump was used. In this case the gain was 27%.

It is also important to make it clear, here, that the introduction of the hump in a Venturi-flume causes a very small effect on the heading up. In the solved example given in (Chapter III) it was found that for a given $h_1 = 1.60 \, \text{ms}$ the discharge was reduced by 4.4% only due to the hump. It was $3.98 \, \text{m}^3/\text{sec.}$, instead of $4.27 \, \text{m}^3/\text{sec.}$, for the horizontal bed.

$$Q = c h^{3/2} \quad \therefore h = k Q^{2/3}$$

$$\therefore h \, dh = \frac{2}{3} k Q^{-1/3} dQ$$

$$dh = \frac{2}{3} dQ/Q = \frac{2}{3} \times 4.4 = 2.93\%$$

It is therefore a good advantage of the hump being a cause of a great reduction in the energy loss which reaches about 55% and in the same time its effect on the heading up does not exceed a small percentage.

KEY OF SYMBOLS GENERALLY USED IN THIS THESIS

Q	discharge.
y	height of the hump.
l	length of the hump.
s	slope of the hump.
d	gate opening.
h	water depth.
$h_{o1} \, h_{o2}$	water depth at the beginning and end of jump.
b	breadth of the flow.
L	length of jump.
H	total energy head.
V	velocity in direction of flow.
v	velocity in horizontal direction.
p	pressure intensity.
P	pressure force.
s	surcharge.
γ	specific weight of water.
ρ_o	density of water.
C	Chezy coefficient.
u	index to bed slope.
m	index to the mean value of slope, velocity, pressure α, α', β . factors.
i	index to inclination of hump.
n	inclination of diverging exit wall.
t	ratio between total Energy head "H" and upstream depth " h_1 ".

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STUDY OF THE JUMP IN A STRAIGHT AND DIVERGENT OPEN CHANNELS

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It is intended in the present work, to study the problem of the hydraulic jump in both cases of the "Standing Wave Weir" and the "Venturi-flume". The first case represents the "hydraulic jump in a *straight* rectangular channel (variable depth), while the second case represents the "hydraulic jump" in a *divergent* rectangular channel (variable width).

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Bidone (1), Bazin (1), Bélanger (1) and Boussinesq (1), failed to verify their analysis largely because they neglected components of pressure on the sloping floor.

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Yarnell (2), carried out an analysis in which floor pressure under the jump were evaluated, assuming straight line profile for the live water from the start of the jump to the end of the roller. This approximation is not accurate enough to lead to satisfactory results.

Carl Kindsvater (3), dealt with this problem on a large scale. He subdivided the common forms of the hydraulic jump in sloping channels into four general groups, depending upon the position of the jump on a sloping floor, his analysis was based on experimental coefficients to evaluate the pressure on the sloping floor (Fig.1). The

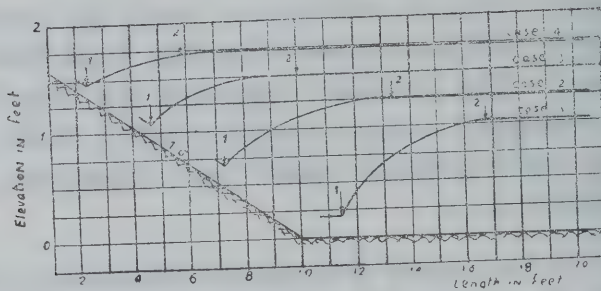


FIG. 1

most important and practical case (Case No. 2) where the toe of the roller is on the slope and the end of the roller is on the horizontal (Case of Standing wave weir), was not studied at all by Kindsvater. Referring to this latter case, he stated "Case 2, is a transition between the first and third cases and neither the equation for the sloping bed nor that for the horizontal one can be applied directly to that case. As shown subsequently, however, he stated, experimental coefficients can be introduced to make the obtained equations applicable to solutions for Case 2". Neither Kindsvater nor the authors who tried to solve the problem, attempted to study the hydraulic jump on general basis.

The Theoretical Treatment

CHAPTER I.

Case of a Straight Channel With a Sloping Bed (Standing Wave Weir)

A.—Limits of the Bed Slope :

For the determination of the extreme limits of the slope, experiments were carried out on a model with a movable plate hinged to a broad crested weir at its crest level. The plate was set at different angles with the horizontal. For each position, tests were carried out at different discharges. Bed pressures at the different points were measured and plotted. It was found that these pressures had the tendency to drop at a certain section which is practically the same for all cases. This tendency increases with the inclination, i.e. the bed pressure decreases with the increase of the inclination, and at an angle 55.5° it becomes negative and hence separation will result. Therefore, there is a particular inclination giving a zero bed pressure at this section. This slope was found to be 42° and was taken as the upper limit for the bed slope.

The inclination is therefore limited to values ranging between zero and 42° .

In order to obtain results covering the whole range between these limits, three different models were used, namely 10° , 21° and 30° degrees. Inclinations less than 10° were excluded, as these are close to the case with horizontal bed. On the other hand, inclination between 30° and 42° were excluded because of the fact that for steep slopes, the overflowing jet cut under the tailwater with little surface disturbance. The hydraulic jump will thus act as submerged or drowned jump or even a submerged outlet, and the high velocity jet extends for a long distance along the floor and hence causes a big scour and requires an expensive long solid floor downstream the structure.

B.—*The Forces Acting upon a Vertical Strip Within the Jump:*

In the case of the hydraulic jump, there are two distinct regions of flow namely : the main live stream, and the eddying dead water (which is usually referred to as “roller”) which remains in place and is located between the toe and the end of the jump. Beneath the roller, the main flow takes place, this is referred to as the “live water”, (Fig. 2).

The mechanism of turbulence will form some sort of momentum carrying system transferring it from one layer to the other. This will be associated with shear stresses of varying intensity with respect to the distance travelled forward by the stream, and continuously reducing the difference between the original velocities of the two streams. The stream surface between the two streams will have on both sides two boundary layers very much similar to that associated with solid boundaries with the exception that the intensity of shear stresses are relatively higher.

The hydraulic jump comprises, however, two problems, each of which is of quite complicated nature, The first is the presence of boundary layers with an adverse pressure gradient along the solid boundaries, and turbulent mixing between two streams, namely the live water and the eddying roller. The stream surface, which is the bordering surface between the live water and eddying roller, will have intense shear stresses acting both upon the adjacent live water and the eddying roller. The shear stresses are obviously caused by momentum transfer carried through the turbulent mixing of eddying water and the live water.

The estimation of these shear stresses and the weight of the live and of the eddying dead water, require in the first place the determination of the profile of the live water surface as well as that of the roller. This can be done with the help of velocity and pressure distribution diagrams.

1. *The lower surface :*

It is the surface between the “live water” and the “eddyng dead water”. For each vertical velocity traverse, (Figs. 2,3), the velocity

WEIR (1)

Q = 35.00 lit./sec.

water surfaces

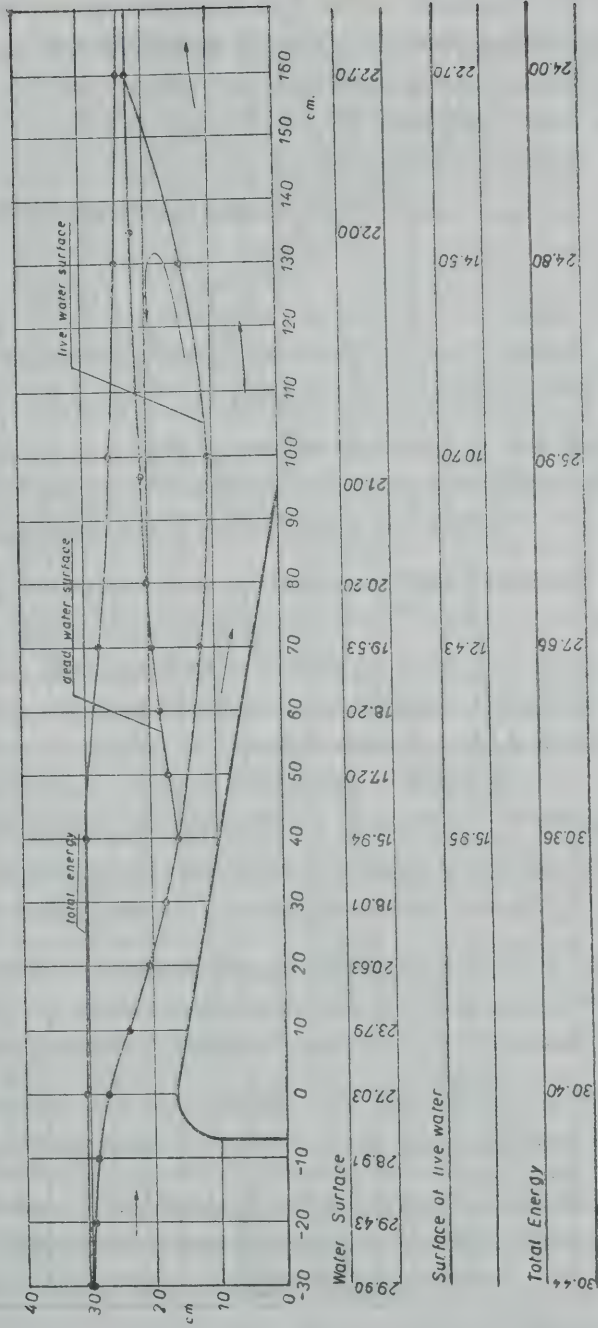


Fig. 2

Scale 1:5

WEIR (1)
Q = 3500 lit/sec.

PRESSURE AND VELOCITY DISTRIBUTION DIAGRAMS AT SECTIONS 70, 100 & 130 cm WITHIN THE JUMP

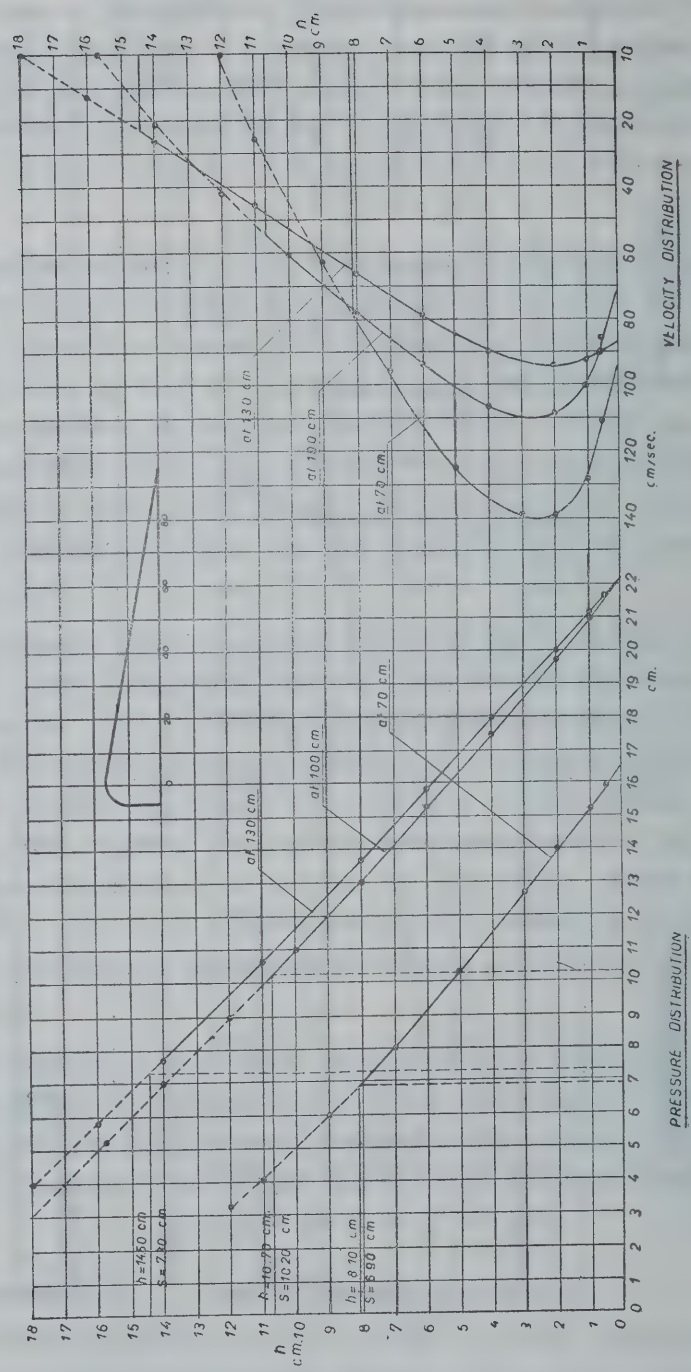


Fig. 3

distribution curve was subdivided into several horizontal strips. The area of each strip was computed and when multiplied by the breadth of the channel, the discharge through the strip could be estimated. This was carried out to plot the discharge per unit width of channel versus depth. Knowing the actual unit discharge value as measured by the 90° V-notch, the limiting surface of the live water could hence be determined accordingly, (Fig. 4) for weir (1).

2. *The upper surface :*

This is the free surface of the eddying dead water. This can be determined by the help of the pressure distribution curves (Fig. 3).

Obviously the unstable nature of that surface and the presence of considerable percentage of entrained air makes it difficult to use direct or visual measurements for determining its profile.

For each vertical section, the heights of the live water were plotted on the pressure distribution diagrams "ab" (Fig. 5). The eddying dead water is considered as a surcharge over the live water. The ordinate of the pressure distribution diagram at the surface of the live water "bc" was assumed to be equivalent to the weight of the dead water column at that section i.e. $bc = bd$. The point "d" is therefore a point on the surface of the eddying dead water, if air bubbles were excluded. Repeating this procedure for other traverses, the hole profile of the upper surface-excluding the entrained air-will be determined.

The deviation between the recorded free surface as determined approximately by visual measurements, "e" (Fig. 5), from that fixed by the above pressure diagram procedure "d", could be used to estimate roughly the percentage of entrained air, i.e. $\frac{ed}{bd}$ is the ratio of the entrained air in weight to the eddying dead water.

The Forces: Considering any arbitrary vertical strip within the jump, 1—2 (Fig. 6), the forces acting upon the control surface enclosing the live water within that strip are as follows:—

Friction forces between live water and solid boundaries of the channel.

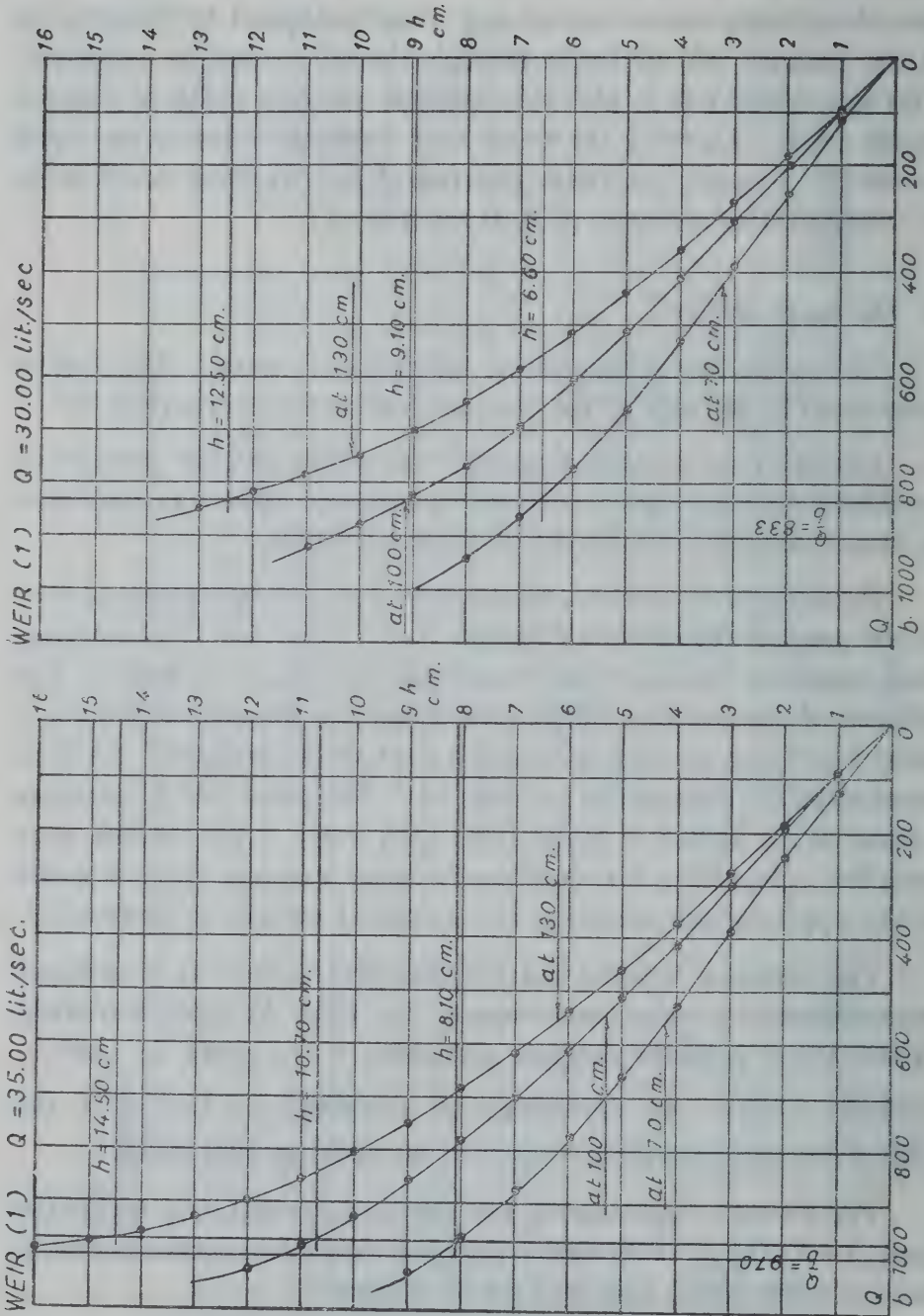
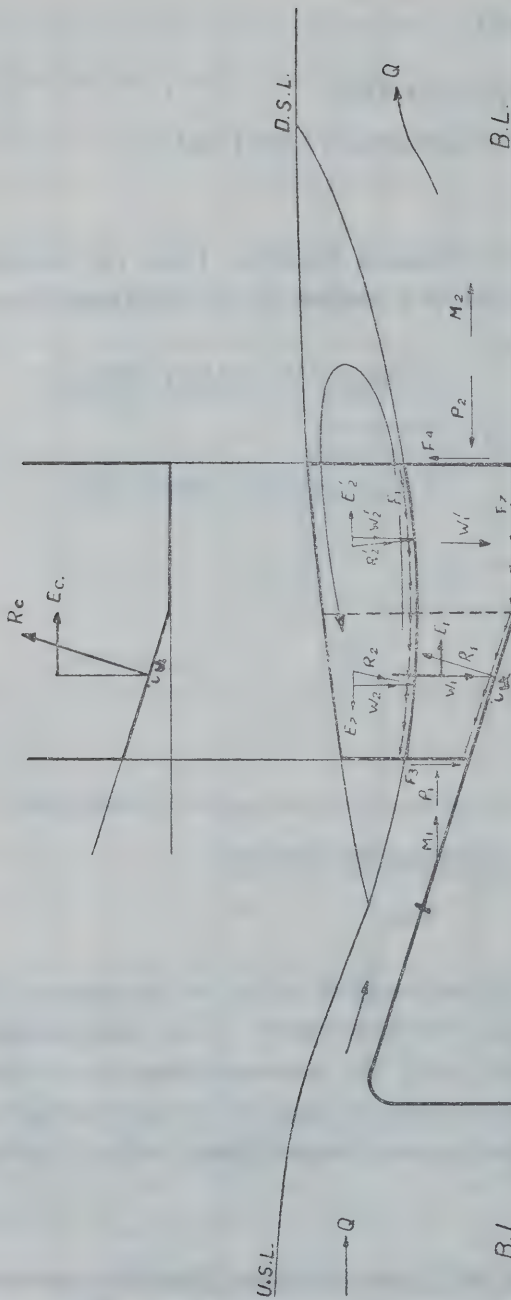


FIG. 4



W = weight of strip , R_C = reaction due to centrifugal forces , F_1 = shearing force due to roller ,
 P = pressure force , R_1 = reaction of bed upon fluid , F_2 = " " " " solid boundaries
 M = momentum /sec. , R_2 = " " roller upon live water , F_3 & F_4 = vertical shearing force.

Forces acting on a strip between two vertical sections within the jump

Fig. 6

pressure along the bed will be higher than its hydrostatic value, (Fig. 7). This force can be estimated with the help of the pressure distribution diagrams obtained for traverses 1—1 and 2—2. Taking the dynamic pressure at the bed as $\left(\frac{\Delta P}{\gamma}\right)_{c1}$ and $\left(\frac{\Delta P}{\gamma}\right)_{c2}$ at traverses 1 and 2 respectively, then the average value will be :

$$\left(\frac{\Delta P}{\gamma}\right)_{c1,2} = \frac{1}{2} \left\{ \left(\frac{\Delta P}{\gamma}\right)_{c1} + \left(\frac{\Delta P}{\gamma}\right)_{c2} \right\}$$

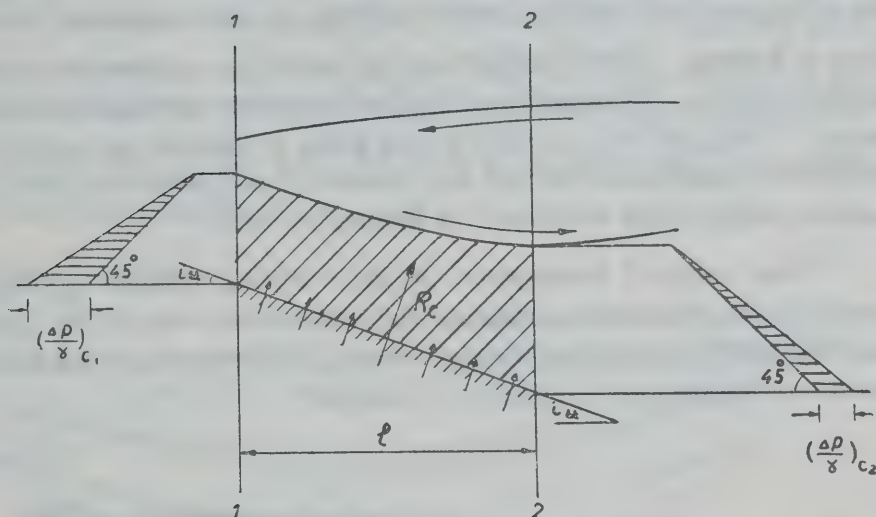


FIG. 7

The total dynamic force acting upon the fluid can be put as :

$$R_c = \gamma \left\{ \frac{\left(\frac{\Delta P}{\gamma}\right)_{c1} + \left(\frac{\Delta P}{\gamma}\right)_{c2}}{2} \right\} \times [l / \cos i_u] \times b.$$

where,

l = horizontal distance between traverses 1 and 2

b = breadth of channel.

These are the forces acting upon the live fluid within the control surface.

Since all the acting forces are seriously influenced by the curvature and inclination of the streamlines, therefore it is justified to emphasise the importance of taking into account this influence when formulating the basic equations used for solving the problem.

C.—The basic equations :

The length of the jump will be determined by taking successive vertical strips. For each strip, the above-mentioned forces will be computed. The basic equations relating these forces will be put in such a way to give the live water depth h_2 at the downstream traverse enclosing the strip, provided that the water depth h_1 at the beginning of the strip is known. Work could then be carried in succession to the following strips. This finally will define the surface profile for the live water. The depth of the eddying dead water at various traverses "s" will also be determined during the process of computations, thus the surface profile of the roller will be defined as well.

If the point of intersection of the resulting profiles O_2 (Fig.8) coincides with the water surface on the downstream side of the jump, then the resulting water profiles are correct, otherwise, the computation has to be repeated till agreement is reached.

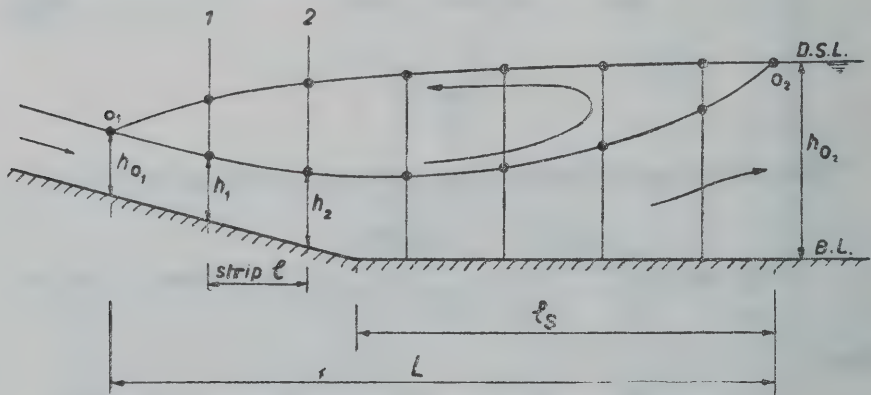


FIG. 8

Approximate estimation of the forces will be made in the first trial making use of such empirical coefficients as obtained from the present experimental work.

Generally speaking, for irrigation channels the normal depth h_{o_2} is known which is practically the water depth at the end of the jump. On the other hand, the water depth h_{o_2} upstream of the jump is usually not given.

Making use of A. Khafagi and S.Z. Abou Hammed work under the title "The curvilinear flow in open channels" (17), one can calculate and draw the water surface over any weir whatever its shape may be, and for any given discharge.

The calculation starts at any arbitrary vertical section across the shooting flow over the weir (Fig. 9). If the point O_2 coincides with the water surface on the downstream side, then the position of the selected section at O_1 is correct. otherwise another section upstream or downstream of O_1 as the case may be, is to be chosen and the calculations are then repeated.

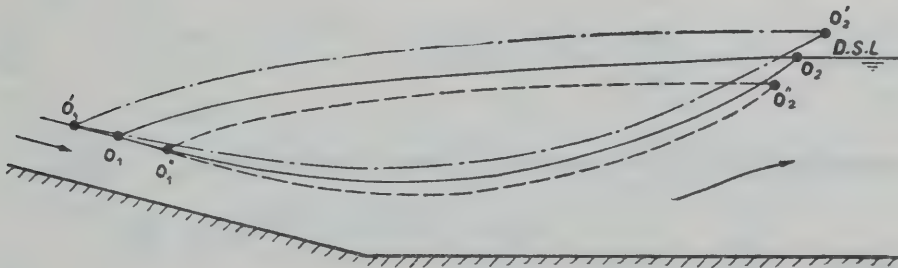


FIG. 9

The equations used: Referring to (Fig. 6), the following equations can be put down :

I.—*The impulse equation :*

Applying the momentum theorem, the horizontal components of the forces acting upon the fluid within the control surface enclosing the live water for any given strip, will be related by the following equation :-

$$P_1 - P_2 + E_1 \pm E_2 + E_c - F'_1 - F'_2 = M_2 - M_1 \quad (1)$$

1.—The pressure forces P_1 and P_2

Taking $(\frac{P}{\gamma} + z)_m$ = the mean pressure head of the live water at any vertical section, “s” = the height of the “eddying dead water” at the considered section excluding the air bubbles, then the pressure distribution diagram in the live water may take the shape shown in (Fig. 10). Then,

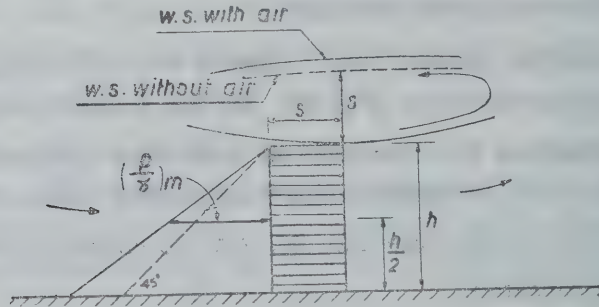


FIG. 10

$$P = \left(\frac{P}{\gamma}\right)_m \times h + s h. \quad (2)$$

assuming that $(\frac{P}{\gamma})_m$ occurs at $\frac{h}{2}$ from the bed, and

$$\text{putting } \left(\frac{P}{\gamma} + z\right)_m + s = D. \quad (3)$$

$$\therefore D = \left(\frac{P}{\gamma}\right)_m + h/2 + s$$

$$\text{or } \left(\frac{P}{\gamma}\right)_m = D - h/2 - s$$

$$\text{Therefore } P = (D - \frac{h}{2} - s) h + s h$$

$$= D h - \frac{h^2}{2} - s h + s h$$

or $P = D h - \frac{h^2}{2}$ (4)

2. The horizontal components E_1 and E_2

$$E_1 = \gamma \left\{ \frac{(h_1 + s_1) + (h_2 + s_2)}{2} \right\} \times l \times b \times \tan i_u \quad (5)$$

Taking “ β ” defined as the ratio of the actual pressure to the hydrostatic pressure, then,

$$\beta = \frac{\left(\frac{P}{\gamma} + z \right)_m \times \frac{h}{2}}{h^2/2} = \frac{\left(\frac{P}{\gamma} + z \right)_m}{h}$$

Substituting this value in equation (3) we get:

$$D = \left(\frac{P}{\gamma} + z \right) + s = \beta h + s$$

or $s = D - \beta h$ (6)

$$\therefore E_1 = \gamma/2 \{ [D_1 - (\beta_1 - 1) h_1] + [D_2 - (\beta_2 - 1) h_2] \}$$

$$l. b. \tan i_u \quad (7)$$

And,

$$\begin{aligned} E_2 &= \gamma \left\{ \frac{s_1 + s_2}{2} \right\} l. b. \tan i_u \\ &= \gamma/2 \{ (D_1 - \beta_1 h_1) + (D_2 - \beta_2 h_2) \} l. b. \tan i_u \quad (8) \end{aligned}$$

3. The horizontal components F'_1 and F'_2

The horizontal component of the shearing force (between the live stream and the roller) is:

$$F'_1 = \tau_s \frac{l b}{\cos i_s} \times \cos i_s = \tau_s l. b.$$

where $\tau_s = \text{shear stress} = \frac{1}{2} \rho_o V_{o1}^2 f_s$

Therefore :

$$F'_1 = \frac{1}{2} \rho_o V_{o1}^2 f_b l b \quad (9)$$

The horizontal component of the frictional force at the bed and sides of the channel F'_2 is :

$$F'_2 = \tau_o A \cos i_u, \text{ where:}$$

τ_o = shear stress at the solid boundary

A = the area of contact of the liver water with the solid boundaries.

$$\text{Putting } \tau_o = \frac{1}{2} \rho_o v_m^2 f_b \times A \cos i_u \quad (10)$$

Using the Chezy's formula $C = \sqrt{\frac{2g}{f_b}}$, f_b can be obtained for the channel.

4. The momenta M_1 and M_2

The momentum $m v_m = \frac{W}{g} v_m$ will be multiplied by a coefficient α' ,

$$\text{where } \alpha' = \frac{\int v^2 dh}{v_m^2 \cdot h}$$

v_m is the horizontal component of the actual mean velocity V_m , i.e.

$$V_m = v_m \sec i_m$$

5. The horizontal force E_c

This is the horizontal component of the force R_c which counts for the effect of the curvature and inclination of the stream lines.

$$\therefore E_c = \gamma/2 \left[\left(\frac{\Delta P}{\gamma} \right)_{o1} + \left(\frac{\Delta P}{\gamma} \right)_{c2} \right] \left\{ \frac{1}{\cos i_u} b \cdot \sin i_u \right\} \quad (11)$$

II.—The total energy head equation (Bernoulli):

$$\Delta Ze = \left\{ \left(\frac{P}{\gamma} + z \right)_{m1} + s_1 + t_1' - \left(\frac{P}{\gamma} + z \right)_{m2} + s_2 + t_2' \right\} + \frac{\alpha_m Q^2}{2g} \left[\frac{h_2^2 \sec^2 i_{m2} - h_1^2 \sec^2 i_{m1}}{b_2 h_1^2 h_2^2} \right] \quad (12)$$

where ΔZe = head loss along the strip

t' = the height of the sloping bed above the horizontal bed at the considered section.

$$\alpha = \text{kinetic energy factor} = \frac{\int V^3 dh}{V_m^3 h}$$

III.—Frictional and Shearing Forces:

The frictional forces take place between the live water and the solid boundaries of the channel while the shearing forces take place at the curved plane between the "live water" and the "dead eddying water".

It is obvious that the surface of the live water is not a free surface. However, we cannot consider the flow along the jump to be analogous to that through a closed conduit. In the latter case, the boundary conditions all around the flow are the same, the area of the cross section is also constant. The frictional stresses and consequently, the coefficient "f" is considered constant all along the conduit.

In our case, however, the conditions are different. Computation of the shear stresses at the upper surface of the live water show that these stresses change from strip to strip along the jump, and consequently the coefficient f_s is variable. Therefore, Darcy's equation for the loss in head due to friction in closed conduits could not be applied for this case.

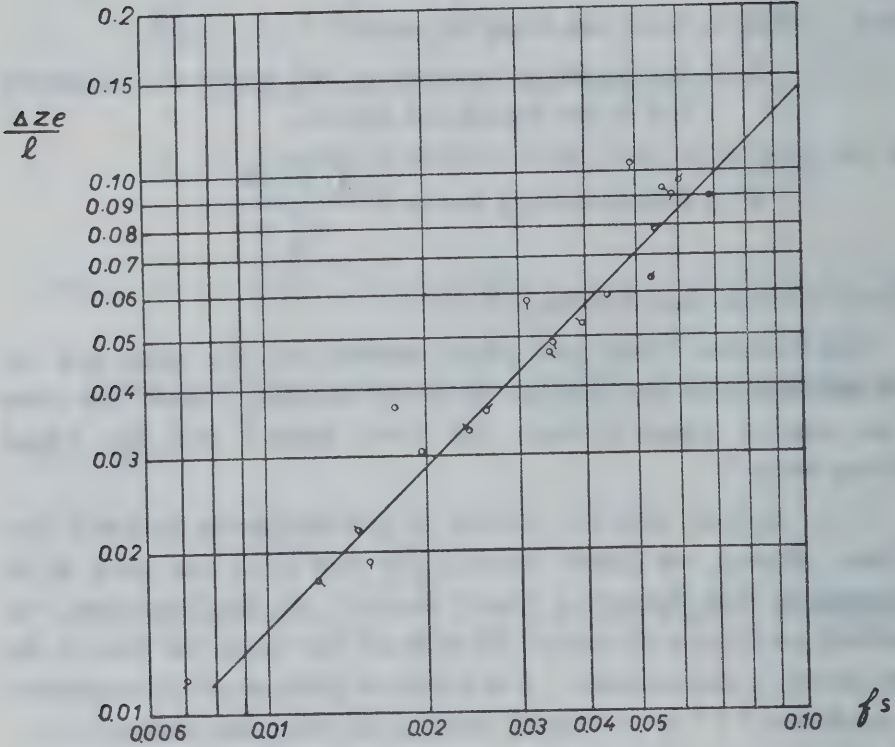
Curves plotted between the virtual slope $\frac{\Delta Ze}{l}$ against f_s (fig.11) give the empirical formula:

$$\frac{\Delta Ze}{l} = K_o f_s^{n_o} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

WEIR (1)

$$\frac{\Delta z_e}{\ell} \rightarrow f_s$$

for all discharges



$$\frac{\Delta z}{\ell} = K_o f_s^{n_o}$$

$$n_o = \frac{8.65}{8.65} = 1.0$$

$$0.05 = k \times (0.035)^{1.0}$$

$$k_o = \frac{0.05}{0.035} = 1.43$$

$$\frac{\Delta z_e}{\ell} = 1.43 f_s^{1.0}$$

o for 35.0 lit./sec

△ " 30.0 "

▽ " 25.0 "

□ " 20.0 "

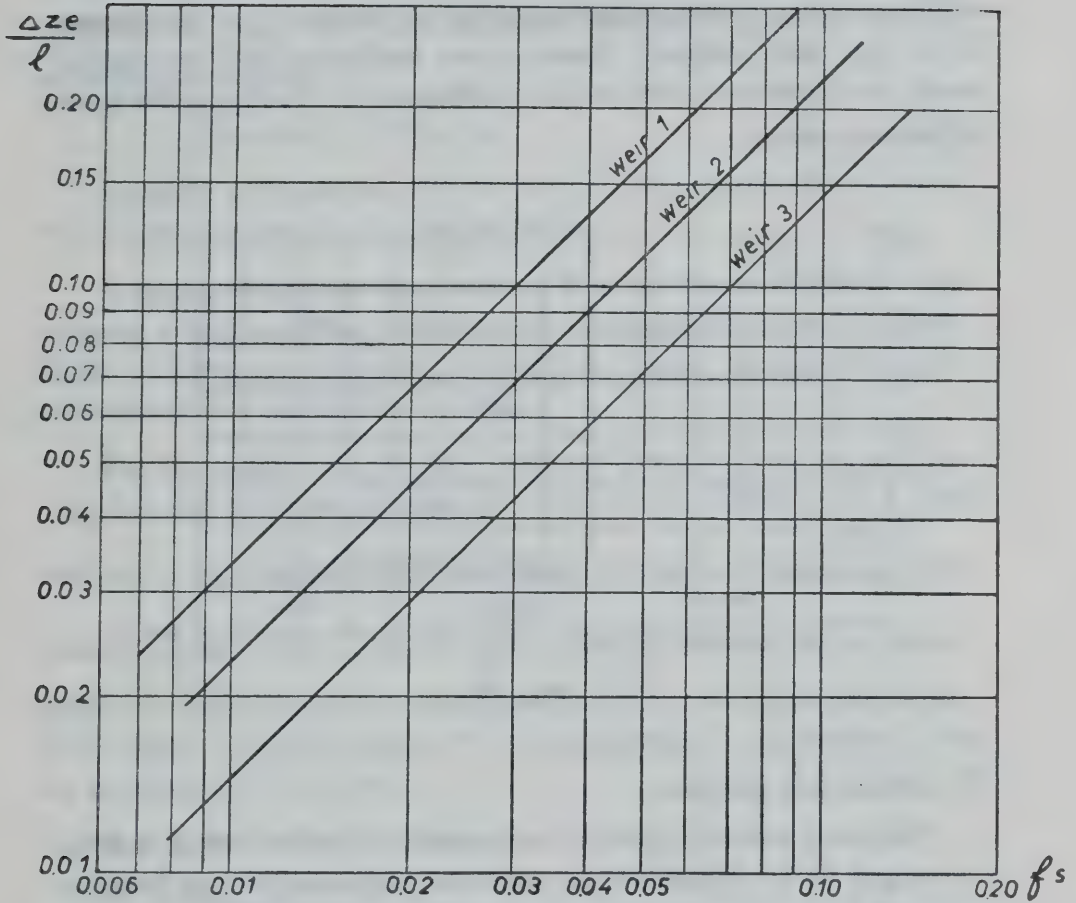
◇ " 15.0 "

× " 10.0 "

FIG. 11 (a)

$$\frac{\Delta z_e}{\ell} \longrightarrow f_s$$

for all weirs



$$\frac{\Delta z_e}{\ell} = K_o f_s^{n_o}$$

$n_o = 1.0$ for all weirs

$K_o = 1.43$ for weir 1

$= 2.26$ „ „ 2

$= 3.25$ „ „ 3

FIG. 11 (b)

where K_0 is a coefficient varying with the slope of the weir (fig. 12) and n_0 is an exponent which is unity for the case of the standing wave weirs.

It is to be noticed, that experiments have shown that friction on the solid boundaries is very small magnitude as compared to shear stresses at the live water surface. Hence it was decided to plot the loss of energy as a function of the friction coefficient " f_s ", as being the main influencing variable.

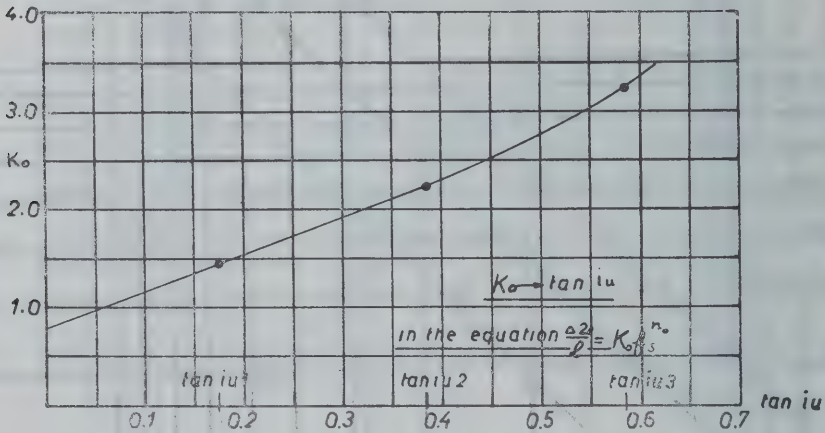


FIG. 12

IV.—Head loss equation :

The head loss ΔZe given in equation (13) is the loss in a strip of a length "1". The head loss from the beginning of the jump at section 0_1 to any section at a distance $\bar{l}$ from 0_1 is $\overline{\Delta Ze}$. The total head loss along the whole length of the jump "L" is $\Sigma \Delta e$.

Curve plotted between $\frac{\overline{\Delta Ze}}{\Sigma \Delta Ze}$ against $\frac{\bar{l}}{L}$ (Fig. 13) gives the following relation:-

$$\frac{\overline{\Delta Ze}}{\Sigma \Delta Ze} + 1.0 = 2.0 \left(\frac{\bar{l}}{L} \right)^{0.25} \quad . \quad . \quad . \quad . \quad (14)$$

This can be put in the form :

$$\overline{\Delta Ze} = \left[\frac{2.0 \sum \Delta Ze}{L^{0.25}} \right] (\bar{I})^{0.25} - \sum \Delta Ze \quad (15)$$

For a given condition, $\sum \Delta Ze = H_{o1} - H_{o2}$, as well as the length of the jump "L", are constant. Therefore equation (15) can be written as follows:

$$\overline{\Delta Ze} = a (\bar{I})^{0.25} - b. \quad (16)$$

In order to determine the factor (a), an approximate value of "L" must be assumed, for the first trial.

Defining the end of the jump as that section at which the flow becomes nearly parallel, Backmateff and matzki determined the functional relationship between the relative length of the jump on a horizontal bed and Froude number.

For the case of the hydraulic jump on a sloping bed or partly on a sloping and partly on a horizontal bed, as in the present work, a more convenient and suitable relation is that between the ratio $\frac{\bar{I}_{toe}}{L}$ and Froude number, (Fig.14) where $\frac{\bar{I}_{toe}}{L}$ is the ratio between the horizontal length of the jump on the sloping part and the total horizontal length of the jump. Froude number "F" corresponds to the section 0_1 at the beginning of the jump.

This relation is found to be:

$$\frac{\bar{I}_{toe}}{L} = C.F. \quad (17)$$

and $C = -0.31 \log_{10} \tan i_u \quad (18)$

For a given discharge passing over a certain standing wave weir, the constants "a" and "b" in equation (16) can therefore be determined.

This equation (16) together with equations (1), (12) and (13) are the four basic equations used for solving the problem.

$$\frac{\overline{\Delta z_e}}{\sum \Delta z_e} \longrightarrow \frac{\bar{\ell}}{L}$$

for all weirs

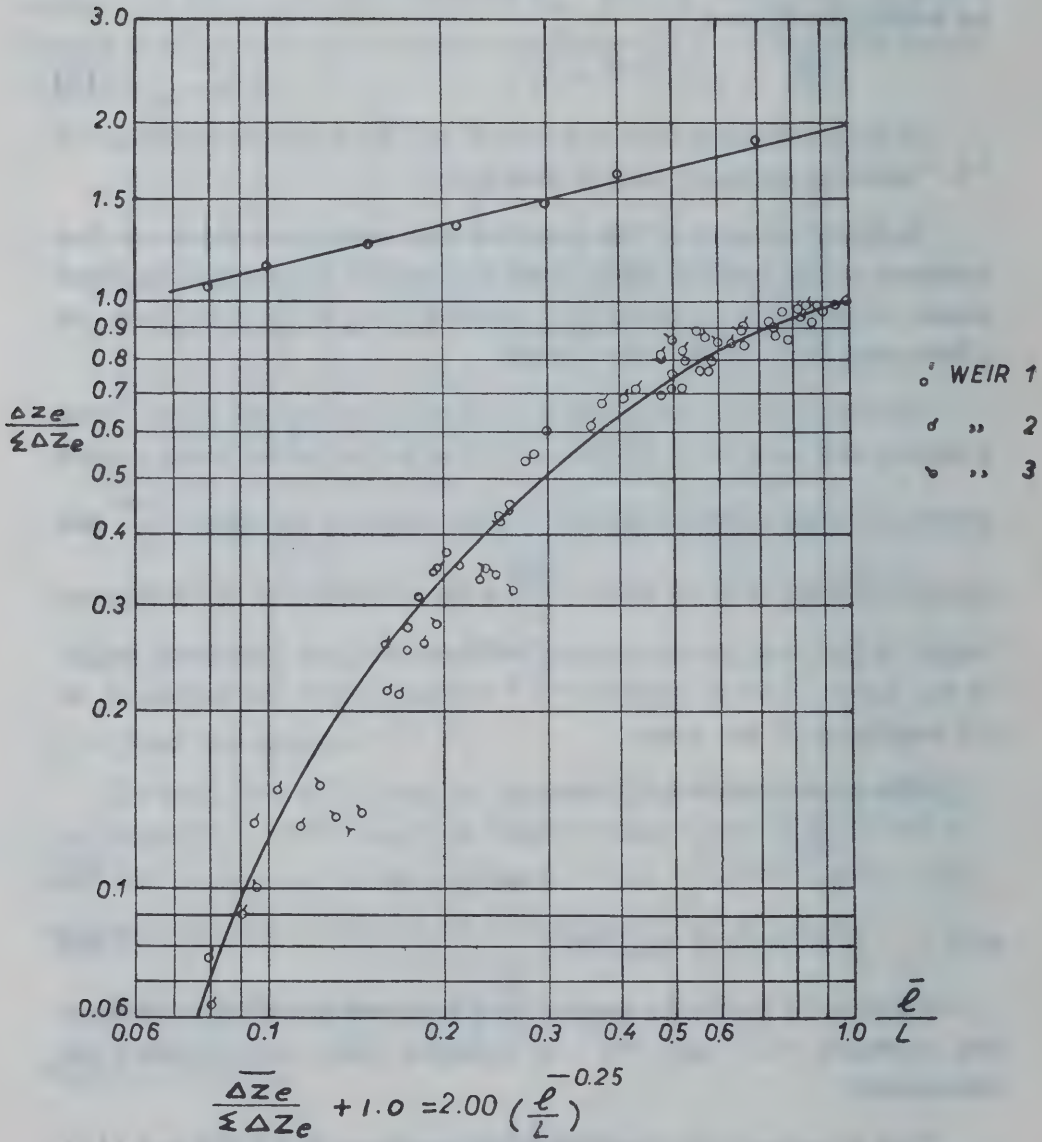


FIG. 13

$$F = \frac{V_{01} \cos i u}{\sqrt{g h_{01}}}$$

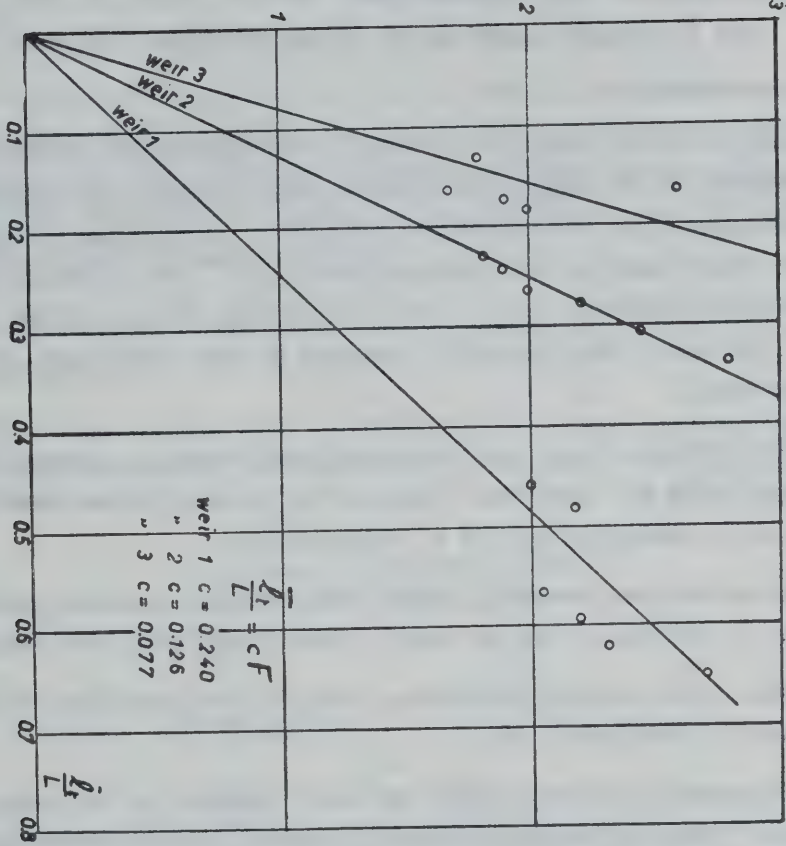


FIG. 14 a

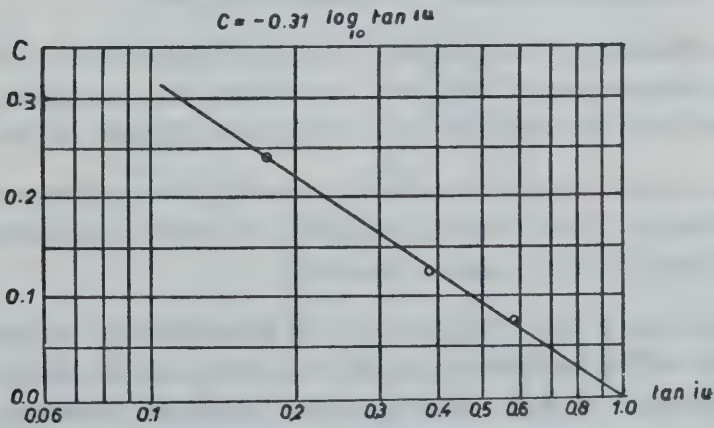


FIG. 14 b

CHAPTER II.

The Hydraulic Jump In A Divergent Open Channel

A.—Introduction :

The hydraulic jump in a straight open channel can be produced downstream of the heading up works such as weirs and regulators. It takes place over the horizontal solid floor. In the case of special type of weirs, namely, the “standing wave weir”, the hydraulic jump occurs either totally or partly over the sloping surface of the weir. This latter case was thoroughly studied in the first part of the present work.

The hydraulic jump in a divergent open channel, *i.e.* channel with divergent sides and horizontal bed, can be produced either totally or partly in the diverging part of a Venturi-flume.

In the first case, however, we deal with a two-dimensional problem, whereas in the second one, we have a three-dimensional flow problem.

Due to the expected difficulties, none, to our knowledge, tried to solve such a complicated case.

Fortunately, the way which we have followed in this work, to solve the “two-dimensional problems”; especially that concerning the “standing wave weir” has led to the main lines for solving the “three-dimensional problem” in a satisfactory way.

The difficulty of the problem is not only because of the divergence but also because part of the jump lies within the divergence and the other part may be outside of it, *i.e.* in the straight part of the channel.

The method which we used for treating the problem overcomes this difficulty. The hydraulic jump will be divided into several strips and each strip will be treated separately.

The effect of the divergence will be introduced in the basic equations. It will be represented by the components of the side pressures, in the direction of flow. An improved theoretical treatment of the water surface in the divergence and upstream the jump was made for

the first time taking into consideration the effect of the divergence on the shape of the water surface. Such calculation of the water surface leads to accurate estimation of the hydraulic jump.

The object of the second part of the work is not only the study of the characterisic and behaviour of the jump in the divergence, but also, and in the first place, to determine the length of the solid floor beyond the throat of the Venturi-flume, for the purpose of getting a convenient and economical design.

B.—Determination of the water surface upstream of the jump in the divergence:

The water surface just upstream of the jump must be known before determination of the dimensions of the jump. In (Fig.15):-

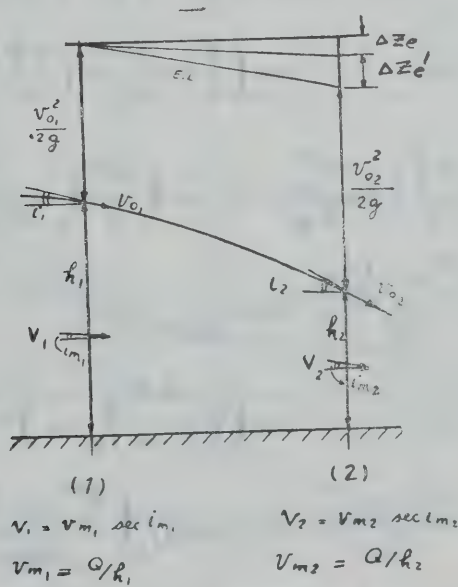


FIG. 15

$$\left(\frac{P}{\gamma} + z \right)_1 + \frac{V_1^2}{2g} = \left(\frac{P}{\gamma} + z \right)_2 + \frac{V_2^2}{2g} + \Delta Z_e + \Delta Z_e'$$

where $V_1 = v_{m1} \sec i_{m1}$ and $V_2 = v_{m2} \sec i_{m2}$

Δz_e = head loss due to friction between section 1 and section 2.

$\Delta z_{e'}$ = head loss due to divergence.

Putting for $\Delta z_{e'}$, the value $(1-\eta) \frac{V_1^2 - V_2^2}{2g}$

$$\text{we get: } \left(\frac{P}{\gamma} + z \right)_1 + \frac{V_1^2}{2g} = \left(\frac{P}{\gamma} + z \right)_2 + \frac{V_2^2}{2g} + (1-\eta) \times \left(\frac{V_1^2 - V_2^2}{2g} \right) + \Delta z_e$$

$$\text{but } h_1 = \left(\frac{P}{\gamma} + z \right)_1 \pm \left(\frac{\Delta P}{\gamma} \right)_1$$

$$\text{and } h_2 = \left(\frac{P}{\gamma} + z \right)_2 \pm \left(\frac{\Delta P}{\gamma} \right)_2$$

$$\therefore h_1 \pm \left(\frac{\Delta P}{\gamma} \right)_1 + \frac{V_1^2}{2g} = h_2 \pm \left(\frac{\Delta P}{\gamma} \right)_2 + \frac{V_2^2}{2g} + (1-\eta) \times \frac{V_1^2 - V_2^2}{2g} + \Delta z_e$$

$$\text{or } (h_1 - h_2) = \left[\left(\pm \frac{\Delta P}{\gamma} \right)_2 - \left(\pm \frac{\Delta P}{\gamma} \right)_1 \right] + \frac{V_2^2 - V_1^2}{2g} + (1-\eta) \left(\frac{V_1^2 - V_2^2}{2g} \right) + \Delta z_e$$

$$\text{but } h_1 - h_2 = \Delta z_w$$

$$\therefore \Delta z_w = \left(\pm \frac{\Delta P}{\gamma} \right)_2 - \left(\pm \frac{\Delta P}{\gamma} \right)_1 + \eta \frac{V_2^2 - V_1^2}{2g} + z_e$$

$$\text{Putting } \left(\pm \frac{\Delta P}{\gamma} \right)_2 - \left(\pm \frac{\Delta P}{\gamma} \right)_1 = \psi \left(\frac{V_2^2 - V_1^2}{2g} \right) \quad (19)$$

$$\therefore \Delta z_w = (\psi + \eta) \left(\frac{V_2^2 - V_1^2}{2g} \right) + \Delta z_e \quad (20)$$

Putting again $(\psi + \eta) = \lambda$.

$$\therefore \Delta z_w = \lambda \left(\frac{V_2^2 - V_1^2}{2g} \right) + \Delta z_e \quad (21)$$

This equation contains the effect of divergence “ η ” as well as the effect of the curvature and inclination of the stream lines “ ψ ”.

Curves between λ and $\frac{\bar{l}_v}{L_v}$ for two different divergence (1:6 and 1:10) were shown in (Fig. 16), where :

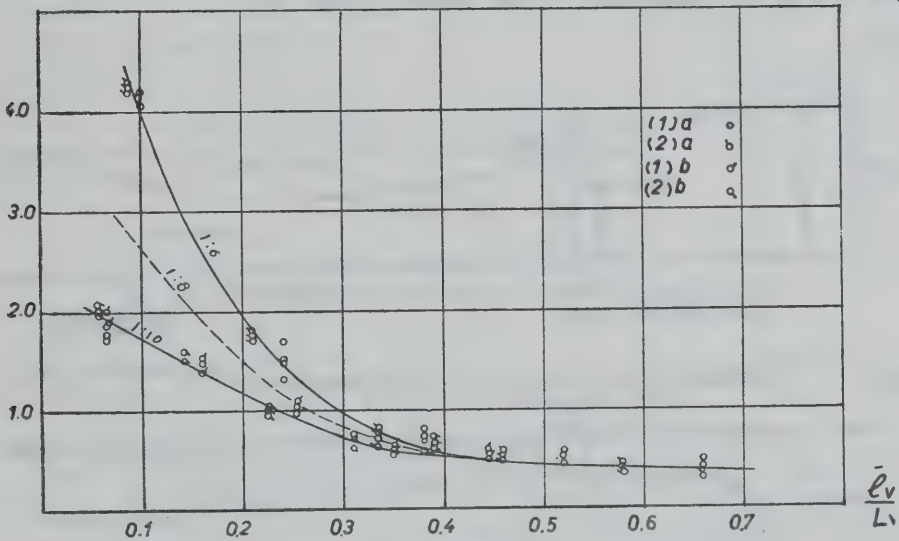


FIG. 16

$\bar{l}_v$ = distance between the throat and the considered section, and

L_v = total length of divergence.

It can be noticed that the difference between the maximum and minimum values of “ λ ”, for the same divergence, will be smaller for small inclination of the sides. This means that if we choose a divergence of about 1:20, say, a mean value for “ λ ” can be used for all sections across the whole length of the divergence.

This theoretical treatment shows how to calculate the water surface in the divergence (upstream of the jump). This calculation cannot be made, unless a point on the water surface is given, from which the calculation begins. The determination of this point was given by A. Khafagi, in his work "Der Venturi-kanal theorie und Anwendung" (7). He proved that in the case of curved streamline flow, the critical depth is not a vertical depth, but a curve which passes by all points where the velocity is equal to the critical velocity.

This curve is a curve with equal velocities, equal $\left(\frac{P}{\gamma} + z\right)$ and perpendicular to the streamlines. It describes (for the case of the Venturi-flume) a parabola. It was assumed that this curve intersects the bed at the beginning of the throat, (Fig. 17).

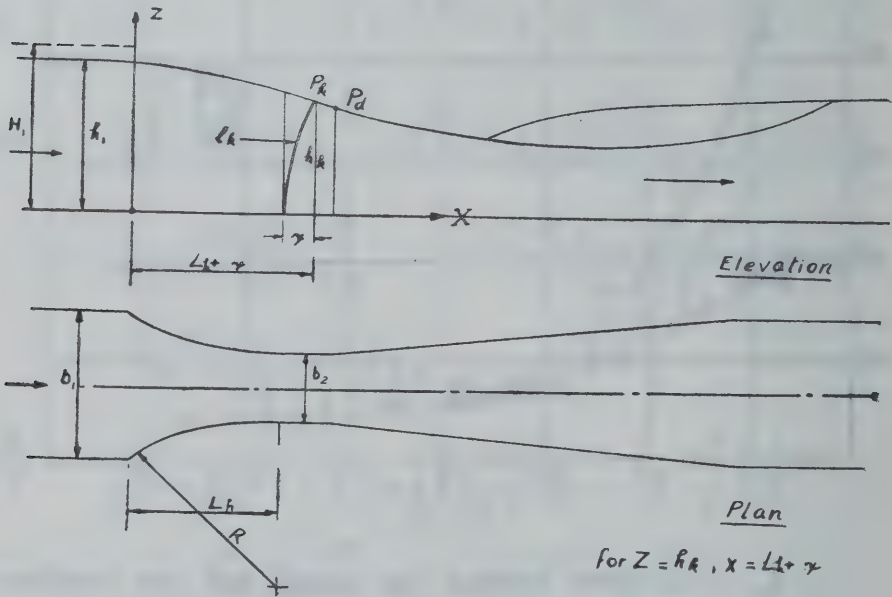


FIG. 17

On these basis, if the discharge is given, the position, shape and length of this curve can be calculated and hence its intersection point with the required water surface is determined.

Starting from this known point, the water surface upstream, as well as downstream of it can be calculated. From the point of

intersection of this water surface with the end of the throat P_d , the calculation of the water surface in the divergene can be begun, according to the previously mentioned theoretical treatment.

The determination of the given point P_d (Fig. 17), follows according to the following equation given by A. Khafagi (7), for a given Venturi-flume with a throat ratio $s = \frac{b_2}{b_1}$, and a given discharge Q :

$$1. \quad 0.148 s^2 t^3 - t + 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$2. \quad H = t h_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

$$3. \quad h_k = 0.73 H = 0.73 t h_1 \quad . \quad . \quad . \quad . \quad . \quad (24)$$

$$4. \quad a = \frac{\sqrt{L_h^2 + (2.93 t - 2.14 t^2) h_1^2} - L_h}{1.07 t^2 h_1^2} \quad . \quad . \quad (25)$$

(a = parameter of the critical curve parabola)

$$5. \quad x = a b_k^2 \text{ (equation of the mentioned parabola) } . \quad (26)$$

$$6. \quad l_k = \frac{1}{4 a} \left[2 a h_k \sqrt{1 + 4 a^2 h_k^2} + \log (2 a h_k + \sqrt{1 + 4 a^2 h_k^2}) \right] \quad . \quad (27)$$

(l_k = length of critical curve)

$$7. \quad z = - \frac{(h_1 - h_2)}{(L_h + x)^2} x^2 + h_1 \quad . \quad . \quad . \quad . \quad (28)$$

(curve of the water surface upstream and inside the throat).

$$8. \quad Q = 2.295 b_2 \sqrt{t h_1} \cdot l_k \quad . \quad . \quad . \quad . \quad (29)$$

C.—Determination of the upper and lower surfaces of the roller, in the divergence:

The application of the theoretical treatment will be different here since the bed of the flume is horizontal and the sides are divergent. In other words, for the weir the flow is more or less two-dimensional, while in case of the flume it is a three-dimensional flow problem. For this reason, the four basic equation given in Chapter I cannot be directly applied for the hydraulic jump in the flume, before they are modified by introducing the effect of divergence.

THE BASIC EQUATIONS

I.—The Impulse Equation :

Figure 18 shows a strip across the jump in the divergence between the vertical sections 1 and 2. The forces acting on the strip are :

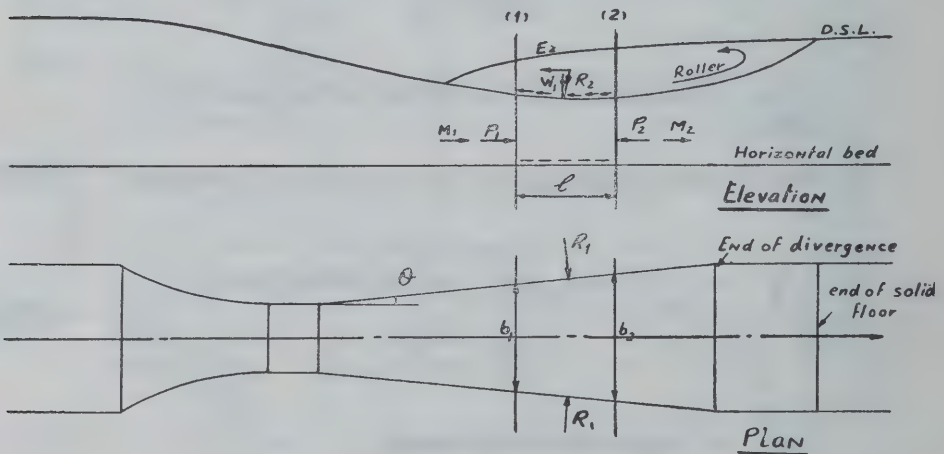


FIG. 18

(a) P_1 and P_2 being the horizontal pressure forces on the upstream and downstream vertical sides of the strip respectively.

(b) E_2 , the horizontal component of the force exerted by the roller upon the moving live water.

(c) F , the horizontal component of the average frictional and shearing forces. (The frictional forces between the live water and the solid boundaries, while the shearing forces between the live water and the roller).

(d) R_1 , the force exerted by one of the diverging sides upon the water within the control surface considered.

Hence: $R = 2R_1 \sin \theta$, where θ is the angle of divergence of one side.

(e) M_1 and M_2 , the horizontal components of the momentum per second entering and leaving the strip.

For equilibrium therefore.

$$P_1 - P_2 + E_2 + R - F' = M_2 - M_1. \quad (30)$$

where:

$$P_1 = (D_1 h_1 - h_1^2/2) b_1, \quad P_2 = (D_2 h_2 - h_2^2/2) b_2 \quad (31)$$

$$E_2 = \gamma/2 \{ (D_1 - \beta_1 h_1) b_1 + (D_2 - \beta_2 h_2) b_2 \} l \tan i_s \quad (32)$$

$$R_1 = \gamma/2 \{ (D_1 h_1 - h_1^2/2) + (D_2 h_2 - h_2^2/2) \} \frac{1}{\cos \theta}$$

$$R = \gamma \{ (D_1 h_1 - h_1^2/2) + (D_2 h_2 - h_2^2/2) \} l \tan \theta. \quad (33)$$

$$F' = \left[\left(\frac{b_1 + b_2}{2} \right) \times 1 \times 2 + \left(\frac{h_1 + h_2}{2} \right) \times 1 \times 2 \right] \tau$$

$$= l (b_1 + b_2 + h_1 + h_2) \frac{1}{2} \rho_o V_{o1}^2 f. \quad (34)$$

where:

τ is the average shear stress,

f is the average friction coefficient, and,

V_{o1} is the mean velocity at beginning of the jump, and

$$M_2 - M_1 = W/g (\alpha'_2 v_{m2} - \alpha'_1 v_{m1})$$

II.—The Total Energy Head Equation:

The structure of this equation remains the same with the only difference that the breadth “ b ” corresponding to each section has to be taken into consideration.

$$\Delta Ze = D_1 - D_2 + \frac{\alpha_m Q^2}{2g} \left[\frac{h_2^2 \sec^2 \alpha_2 b_2^2 - h_1^2 \sec^2 \alpha_1 b_1^2}{b_1^2 b_2^2 h_1^2 h_2^2} \right]. \quad (35)$$

III.—The Frictional and Shearing Force Equation:

The experimental work carried out on flumes with throat ratios of 0.395 and 0.455, and with divergence of 1:6 and 1:10 for each throat ratio, have shown that the relation $\Delta Ze/l = K_o f^{n_o}$. (36) is true for all mentioned flumes, where:

ΔZe is the loss in head along a strip, and

$$f = \frac{\tau}{\frac{1}{2} \rho_o V_{o1}^2}.$$

The exponent “ n_o ” was found to be the same for all cases and equals 0.75. The constant K_o , however, was found to vary with the divergence as well as, with the throat ratio, (Fig. 19). Substituting the value of “ f ” from equation (36) in equation (34) we get:

$$F_1 = l (b_1 + b_2 + h_1 + h_2) \frac{1}{2} \rho_o V_{o1}^2 \left(\frac{\Delta Ze}{l K_o} \right)^{\frac{1}{n_o}}. \quad (37)$$

which can then be substituted in the Impulse equation (30).

IV.—The Loss in Head Equation:

Here also, the dimensionless quantities $\frac{\Delta Ze}{\Sigma \Delta Ze}$ are plotted against $\frac{l}{L}$ for all the tested Venturiflumes and it has been found that all the

points be very close to one single curve, (Fig. 19), the equation for which can be put in the form :

$$\frac{\overline{\Delta Z e}}{\Sigma \Delta Z e} + 1.0 = 2.0 \left(\frac{\bar{1}}{L} \right)^{0.30} \quad . \quad . \quad . \quad . \quad (38)$$

where : $\overline{\Delta Z e}$ = head loss in a distance from the section at the beginning of the jump.

$\Sigma \Delta Z e$ = the total head loss in the whole length of the jump "L".

This means that the loss in head equation is neither affected by the divergence nor by the throat ratio.

Comparing this equation (38) with equation (14) of the weir having parallel sides, a small difference can be noticed in the exponent, which is equal to 0.30 in equation (38) while it is equal to 0.25 in equation (14). The rest of the items for both equations remains the same.

The application of the four mentioned basic equations for successive strips, from the beginning of the jump onwards, leads to the determination of the depths, h_1 , h_2 , D_1 and D_2 for each strip. Thus the surfaces for the live and dead water can be determined.

The intersection of both curves determines the end of the jump.

If the point of intersection lies on the downstream water surface of the original flow in the channel, this means that the calculated curves are real surfaces. If not, the whole calculations must be repeated till that point lies on the downstream water surface.

The mentioned procedure of calculation requires that an approximate value of the length of the jump "L" is to be known. An approximate value for "L" can be determined from a relation between

Froude number "F" and the ratio $\frac{L}{h_{o2}}$, (Fig. 21), where :

h_{o2} = downstream water depth in the original channel, and

$$F = v_{o1} / \sqrt{g h_{o1}}$$

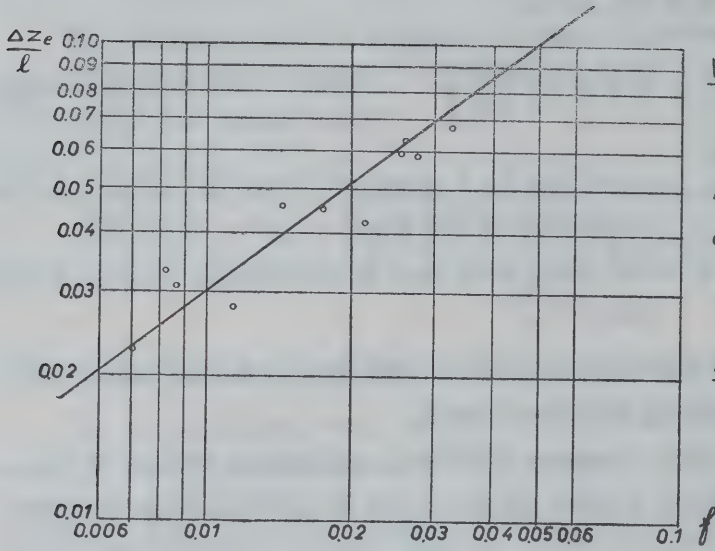


FIG. 19 a

V.F. (1) b

$$\frac{\Delta Z_e}{l} K_o f^{n_o}$$

$$n_o = 0.75$$

$$0.08 = K_o \times (0.036)^{0.75}$$

$$= K_o \times 0.0825$$

$$\therefore K_o = 0.97$$

$$\frac{\Delta Z_e}{l} = 0.97 f^{0.75}$$

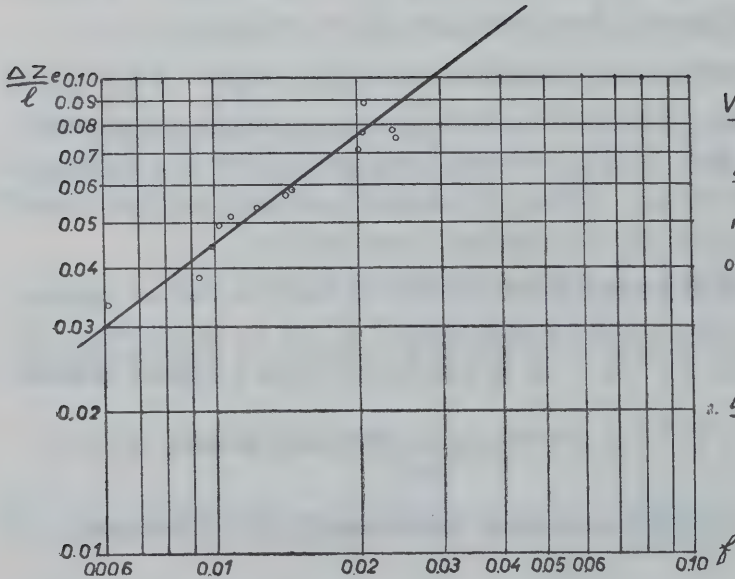


FIG. 19 b

V.F. (1) a

$$\frac{\Delta Z_e}{l} = K_o f^{n_o}$$

$$n_o = 0.75$$

$$0.08 = K_o \times (0.022)^{0.75}$$

$$= K_o \times 0.057$$

$$K_o = 1.4$$

$$\therefore \frac{\Delta Z_e}{l} = 1.4 f^{0.75}$$

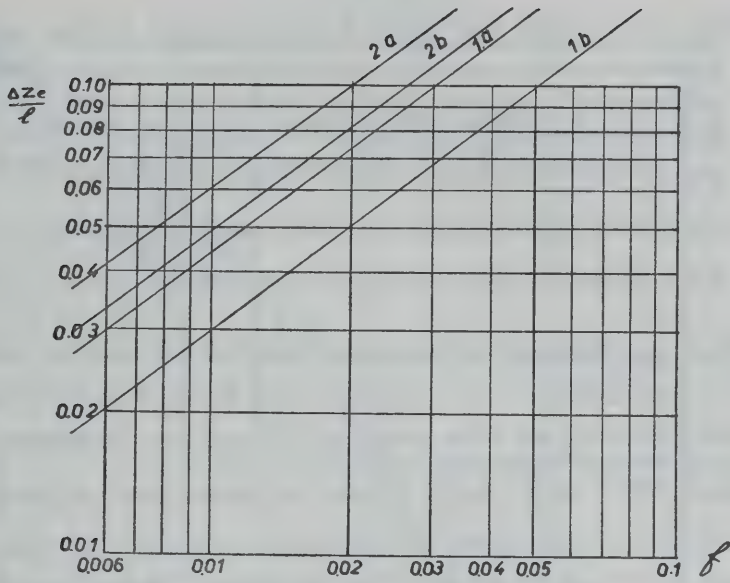


FIG. 19 e

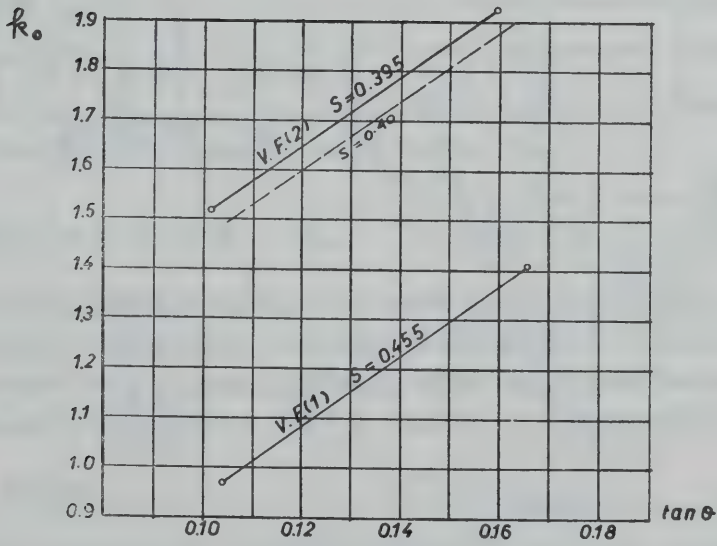
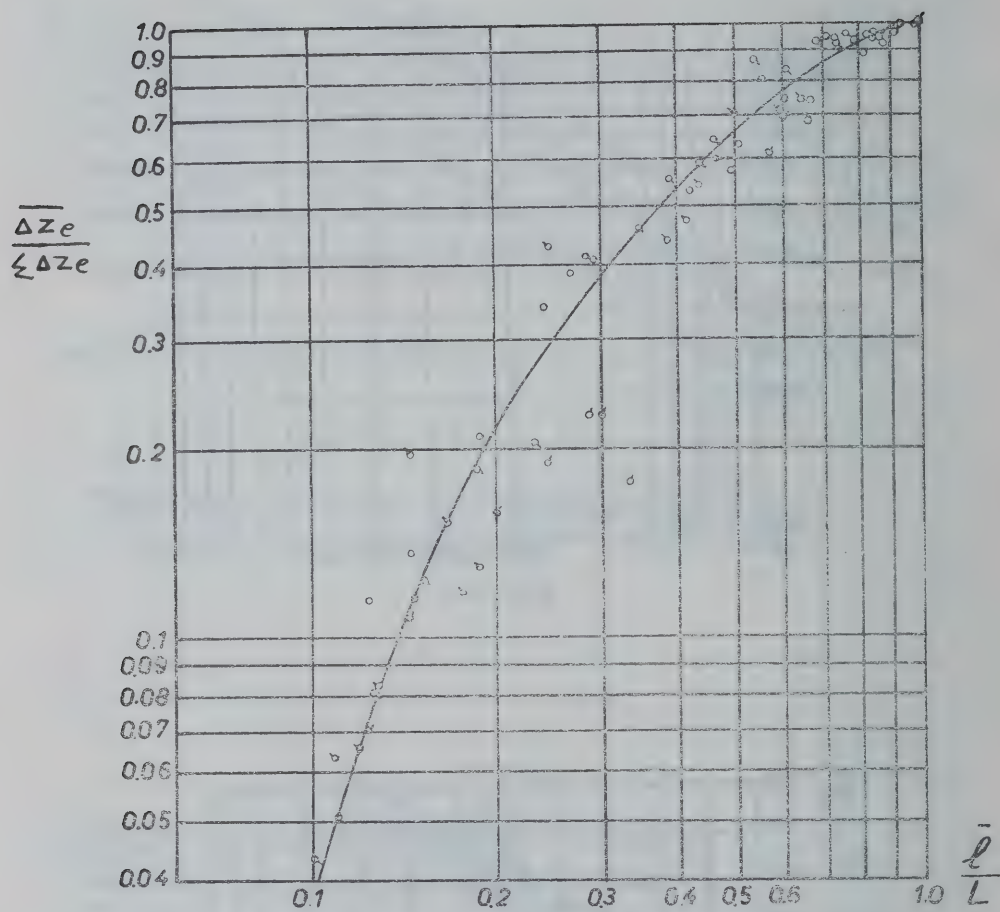


FIG. 19 f



$$\left(\frac{\overline{\Delta Ze}}{\sum \Delta Ze} + 1.0 \right) = K_0 \left(\frac{\bar{\ell}}{L} \right)^{n_0}$$

$$n_0 = \frac{3.30}{10.95} = 0.30$$

$$(0.39 + 1.0) = K_0 \times (0.3)^{0.30}$$

$$\therefore 1.39 = K_0 \times \frac{1.39}{1.995} = K_0 \times 0.697$$

$$\therefore K_0 = 2.00$$

$$\left(\frac{\overline{\Delta Ze}}{\sum \Delta Ze} + 1.0 \right) = 2.00 \left(\frac{\bar{\ell}}{L} \right)^{0.30}$$

1 a

2 a

1 b

2 b

a

a

b

a

FIG. 20

REMARKS

Four factors namely α , α' , β and β' are used in the theoretical treatment :

(a) “ α ” is used as a correction factor for the true kinetic energy head, *i.e.* $\alpha V_m^2/2g$ instead of $V_m^2/2g$, where :

$$\alpha = \frac{\int \int V^3 da}{V_m^2 A}$$

This coefficient was calculated and plotted against $\frac{1}{L}$ at different traverses for the tested rates of flow. It is found that all points follow practically one curve. It starts with unity at the beginning of the jump and reaches the value of 1.42 at $\frac{1}{L} 1.00$. This shows that the velocity distribution at the end of the jump is by no means uniform. It requires a further distance on the downstream side till it reaches its normal state where $\alpha \cong 1.00$.

(b) Another factor $\alpha' = \frac{\int \int v^2 da}{v_m^2 A}$ has to be multiplied by the impulse M .

$$\therefore M = \alpha' m v_m, \text{ where } m = \frac{w}{g} Q$$

It has the same character as the “ α ” curve.

(c) Another factor $\beta = \frac{P_o + P}{P_o}$, in which ΔP is the excess in pressure over the hydrostatic, which is due to the effect of the curvature and inclination of the upper boundary at any section. The inclination of the stream lines causes a decrease, whereas the curvature causes an increase in the pressure. “ β ” is a criterion of both effects. It is plotted against $\frac{1}{L}$.

(d) β' corresponds to the resultant of the side pressure R . β and β' are practically the same.

CHAPTER III

Solved Examples

A.—Standing Wave Weir :

A standing wave weir of the shape shown in (Fig. 21), is constructed across a rectangular channel whose discharge is 62 cubic meter per second. It is required to calculate the scour length l_s , downstream the toe of the weir. The channel is 20.0 m. breadth with a bed slope of 15 cms. pe. kilometer.

Calculations: The normal depth h_{o2} in the channel can be obtained from the formula :

$$Q = A \times \frac{1}{n} m^{\frac{2}{3}} i^{\frac{1}{2}}, \text{ assuming } \frac{1}{n} = 50$$

$$\therefore h_{o1} = \underline{2.90 \text{ m.}}$$

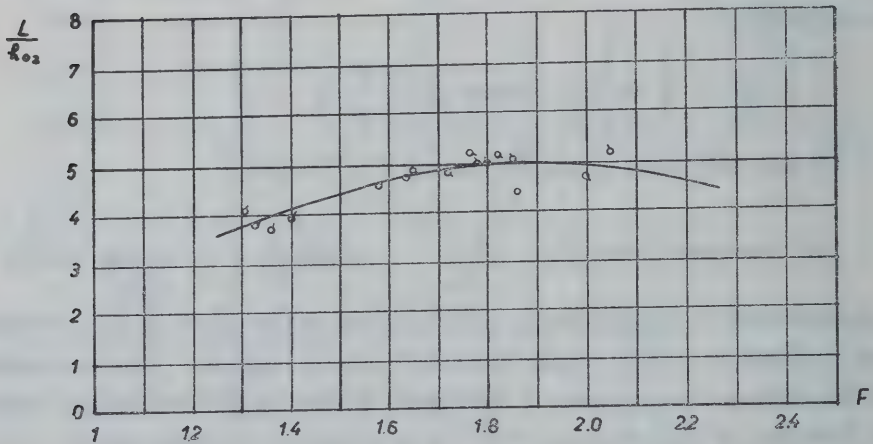


FIG. 21

Making use of A. Khafagi and S. Hammad work (17), the water surface over the weir is calculated and plotted on (Fig. 21).

DETERMINATION OF THE JUMP

First Trial

Assume the jump starts at distance 2.40 m. from 0, where $h_{o1} = 0.61$ m

$$\therefore V_{o1} = Q/b h_{o1} \cos i_u = 5.40 \text{ m/sec.}$$

$$F = V_{o1} \cos i_u / \sqrt{g h_{o1}}$$

Applying equation (18), $C = 0.136$

$$\text{Applying equation (17), } \frac{\bar{l}_t}{L} = cF = 0.285$$

$$\text{since } \bar{l}_t = 4.60 \text{ m., } \therefore L = \underline{16.20 \text{ m.}} \quad \therefore l_s = \underline{11.60 \text{ m.}}$$

Total energy head at beginning of jump " H_{o1} "

$$H_{o1} = t' + \left(\frac{P}{\gamma} + z \right)_m + \alpha \frac{V_{o1}^2}{2g}$$

$$\text{at } \frac{\bar{l}}{L} = 0.00, \beta = 1.22 \text{ and } \alpha = 1.00$$

$$\therefore P_1 = 0.226 \text{ tons, } \therefore \left(\frac{P}{\gamma} + z \right)_m = 0.675 \text{ m.}$$

$$\therefore H_{o1} = 1.675 + 0.675 + 1.492 = 3.842 \text{ m.}$$

Total energy head at end of jump " H_{o2} "

$$\text{at } \frac{\bar{l}}{L} = 1.00, \beta = 1.00 \text{ and } \alpha = 1.40$$

$$\therefore \left(\frac{P}{\gamma} + z \right) = 2.90 \text{ m, } \alpha \frac{V_{o2}^2}{2g} = 0.082$$

$$\therefore H_{o2} = 2.90 + 0.082 = 2.982.$$

$$\therefore \Sigma \Delta Z e = 3.842 - 2.982 = 0.860 \text{ m.}$$

$$\text{Strip No. 1: 2.00 m. length, } \frac{\bar{1}}{L} = 0.124$$

$$\beta_1 = 1.22, \alpha_1 = 1.00, \alpha'_1 = 1.00$$

$$\beta_2 = 1.27, \alpha_2 = 1.02, \alpha'_2 = 1.01$$

from equation (14), $\overline{\Delta Z e} = 0.155 \text{ m.}$

Applying the total energy head equation (12):

$$0.155 = 2.35 - (D_2 + 0.95) + 1.34 \left(\frac{h_2^2 - 0.373}{h_2^2} \right)$$

For arbitrary values of h_2 , this equation is used to find the corresponding values of D_2

$$h_2 \text{ m } 0.60 \quad 0.70 \quad 0.80 \quad 0.90$$

$$D_2 \text{ m } 1.196 \quad 1.565 \quad 1.808 \quad 1.970$$

Applying the impulse equation (1):

$$P_1 = 4.52 \text{ tons, } P_2 = (D_2 h_2 - h_2^2/2) 20 \text{ tons (equation 4)}$$

$$E_1 = (D_2 + 0.61 - 0.27 h_2) 7.28 \text{ tons (equation 5)}$$

$$E_2 = (D_2 - 1.27 h_2) 20 \tan i_s \text{ (equation 8)}$$

$$E_c = \left\{ \left(\frac{\Delta P}{\gamma} \right)_{c_1} + \left(\frac{\Delta P}{\gamma} \right)_{c_2} \right\} 6.84 \text{ tons (equation 11)}$$

$$F'_1 = \frac{1}{2} \rho_o V_{o1}^2 f_s \cdot l. b \text{ and } \frac{\Delta Z e}{l} = 2.18 f_s$$

$$\therefore F'_1 = 2.12 \text{ tons, (equation 9 and 13)}$$

$$f_b = 0.0062$$

$$F'_2 = \frac{1}{2} \rho_o V_m^2 \times 0.0062 [b l + (h_1 + h_2) l] \quad 0.94 \text{ tons (equ.10)}$$

$$M_2 - M_1 = 6.31 (3.14/h_2 - 5.09) \text{ tons}$$

Again, this equation is used to find the values of D_2 corresponding to the selected values of h_2 . Values of E_c are computed separately due to the inclination of the bed and the inclination and curvature of the surface of live water.

h_2 m	0.60	0.70	0.80	0.90
E_c tons	1.38	1.30	1.73	1.92
D_2 m	1.24	1.55	1.79	1.94

Plotting the values of h_2 and D_2 in the two tables we get :

$$h_2 = 0.64 \text{ m, and } D_2 = 1.36 \text{ m, } \therefore s = 0.55 \text{ m.}$$

$$\text{Strip No. 2: 2.00 m length, } \frac{\bar{I}}{L} = 0.284.$$

$$\beta_1 = 1.27, \alpha_1 = 1.02, \alpha'_1 = 1.01$$

$$\beta_2 = 1.15, \alpha_2 = 1.05, \alpha'_2 = 1.015$$

from equation (14), $\overline{\Delta Ze} = 0.395 \text{ m.}$

$$\therefore \Delta Ze \text{ for the 2nd strip} = 0.240 \text{ m.}$$

Applying the total energy head equation 12 :

$$\therefore 0.240 = 2.31 - D_2 + 1.23 \frac{h_2^2 - 0.41}{h_2^2}$$

h_2 m	0.70	0.80	0.90	1.00
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D_2 m	2.270	2.510	2.680	2.797
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Applying the impulse equation (1), we get :

$$h_2 \text{ m} \quad 0.70 \quad 0.80 \quad 0.90 \quad 1.00$$

$$E_c \text{ m} \quad 1.69 \quad 1.31 \quad 1.42 \quad 1.73$$

$$D_2 \text{ m} \quad 2.29 \quad 2.47 \quad 2.625 \quad 2.755$$

Plotting values of h_2 and D_2 in the two tables we get :

$$h_2 = 0.72 \text{ m} \quad \text{and} \quad D_2 = 2.33 \text{ m} \quad \therefore \quad s = 1.50 \text{ m.}$$

Strip No. (3): 3.00 m length, $\frac{1}{L} = 0.469$

$$\beta_1 = 1.15, \alpha_1 = 1.05, \alpha'_1 = 1.015$$

$$\beta_2 = 1.05, \alpha_2 = 1.09, \alpha'_1 = 1.025$$

from equation (14), $\overline{\Delta Ze} = 0.560 \text{ m.}$

$$\therefore \quad \Delta Ze \text{ for the 3rd strip} = 0.165 \text{ m}$$

Applying the total energy head equation 12:

$$\therefore \quad 0.165 = 2.33 - D_2 + 1.005 \left(\frac{h_2^2 - 0.52}{h_2^2} \right)$$

$$h_2 \text{ m} \quad 0.70 \quad 0.80 \quad 0.90 \quad 1.00$$

$$D_2 \text{ m} \quad 2.103 \quad 2.353 \quad 2.525 \quad 2.647$$

Applying the impulse equation (1), we get :

$$h_2 \text{ m} \quad 0.70 \quad 0.80 \quad 0.90 \quad 1.00$$

$$D_2 \text{ m} \quad 2.14 \quad 2.36 \quad 2.493 \quad 2.60$$

Plotting values of h_2 and D_2 in the two tables we get :

$$h_2 = 0.82 \text{ m, and } D_2 = 2.39 \text{ m, } \therefore \quad s = 1.51 \text{ m.}$$

Strip No. (4): 3.00 m length, $\frac{\bar{I}}{L} = 0.655$

$$\beta_1 = 1.05, \alpha_1 = 1.09, \alpha'_1 = 1.025$$

$$\beta_2 = 1.03, \alpha_2 = 1.20, \alpha'_2 = 1.05$$

from equation (14), $\overline{\Delta Z_e} = 0.69$ m.

$\therefore \Delta Z_e$ for the 4th strip = 0.130 m.

Applying the total energy head equation 12 :

$$\therefore 0.130 = 2.39 - D_2 + 0.835 \left(\frac{h_2^2 - 0.672}{h_2^2} \right)$$

$$h_2 \text{ m} \quad 0.90 \quad 1.00 \quad 1.10 \quad 1.20$$

$$D_2 \text{ m} \quad 2.402 \quad 2.534 \quad 2.630 \quad 2.705$$

Applying the impulse equation (1)

$$h_2 \text{ m} \quad 0.90 \quad 1.00 \quad 1.10 \quad 1.20$$

$$D_2 \text{ m} \quad 2.430 \quad 2.55 \quad 2.605 \quad 2.640$$

Plotting values of h_2 and D_2 in the two tables, we get :

$$h_2 = 1.05 \text{ m} , \text{ and } D_2 = 2.585 \text{ m} , \therefore s = 1.505 \text{ m}.$$

Strip No. 5: 3.00 m length, $\frac{\bar{I}}{L} = 0.84$

$$\beta_1 = 1.03 , \alpha_1 = 1.20 , \alpha'_1 = 1.05$$

$$\beta_2 = 1.01 , \alpha_2 = 1.36 , \alpha'_2 = 1.14$$

from equation (14), $\overline{\Delta Z_e} = 0.775$ m.

$\therefore \Delta Z_e$ for the 5th strip = 0.085 m.

Applying the total energy head equation 12:

$$0.085 = 2.585 - D_2 + 0.77 \left(\frac{h_2^2 - 1.10}{h_2^2} \right)$$

h_2 m	1.20	1.40	1.50	1.60
D_2 m	2.634	2.75	2.79	2.823

Applying the impulse equation 1:

h_2 m	1.20	1.40	1.50	1.60
D_2 m	2.65	2.75	2.76	5.78

Plotting values of h_2 and D_2 in the two tables, we get:

$$h_2 = 1.40 \text{ m}, \text{ and } D_2 = 2.75 \text{ m}, \therefore s = 1.33 \text{ m}.$$

Strip No. 6: 2.40 m. length, $\frac{\bar{1}}{L} = 0.96$

$$\beta_1 = 1.01, \alpha_1 = 1.36, \alpha'_1 = 1.14$$

$$\beta_2 = 1.00, \alpha_2 = 1.40, \alpha'_2 = 1.21$$

from equation (14), $\overline{\Delta Ze} = 0.84 \text{ m}.$

$\therefore \Delta Ze$ for the 6th strip = 0.065 m.

Applying the total energy head equation 12:

$$0.065 = 2.75 - D_2 + 0.34 \left(\frac{h_2^2 - 1.96}{h_2^2} \right)$$

h_2 m	2.00	2.20	2.40	2.60
D_2 m	2.858	2.886	2.910	2.925

Applying the impulse equation 1

h_2 m	2.00	2.20	2.40	2.60
D_2 m	2.865	2.89	2.90	2.905

Plotting values of h_2 and D_2 in the two tables, we get :

$$h_2 = 2.30 \text{ m}, \text{ and } D_2 = 2.895 \text{ m}, \therefore s = 0.595 \text{ m.}$$

The point of intersection of the live water and dead water profiles does not coincide with the water surface on the downstream of the jump.

$\therefore$ The computations are repeated again assuming the beginning of the jump at another point on the weir.

Second Trial: Assume the jump starts at distance 2.80 m, from 0, The repeated calculations lead to the result that the point of intersection of the live and dead water profiles, (Fig. 22), occurs at 19.10 m from 0, at $h_{02} = 2.90 \text{ m}$, i.e. coincides with the water surface on the downstream of the jump.

$\therefore$ The correct position of the start of the jump is at 2.80 m from 0, and it ends at 19.10 m.

$$\therefore L = 19.10 - 2.80 = \underline{16.30 \text{ m.}}$$

Hence, $l_s = 19.10 - 7.0 = \underline{12.10 \text{ m.}}$, which is the required length of the solid floor downstream the toe of the weir.

B.—A Venturi-Flume :

Venturi-flume of the type shown in (Fig. 23), is constructed along a rectangular channel the discharge of which is 5.55 cubic meter per second, its breadth 3.00 m, throat breadth of the Venturi-flume 1.20 m, side walls divergence 1 : 8, channel bed slope 15 cm. per km., and downstream water depth in the channel is 1.53 m. It is required to estimate the length of the jump.

CALCULATIONS

Determination of the water surface upstream and inside the throat of the Venturi-flume :

$$S = \frac{b_2}{b_1} = 0.40$$

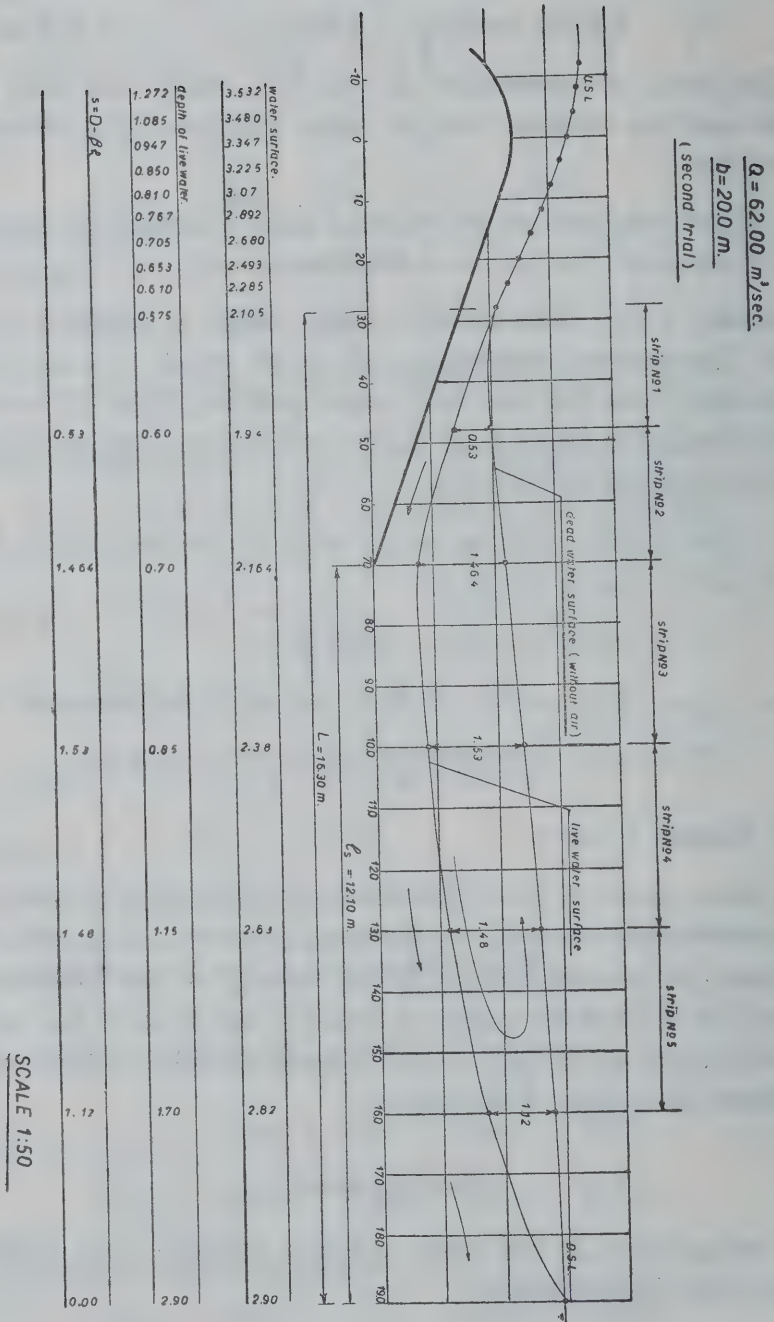


FIG. 22

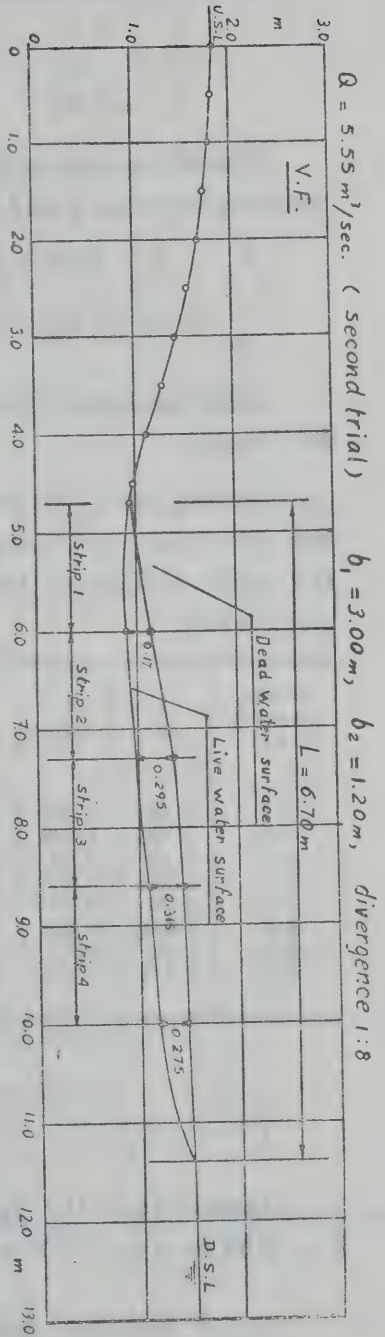
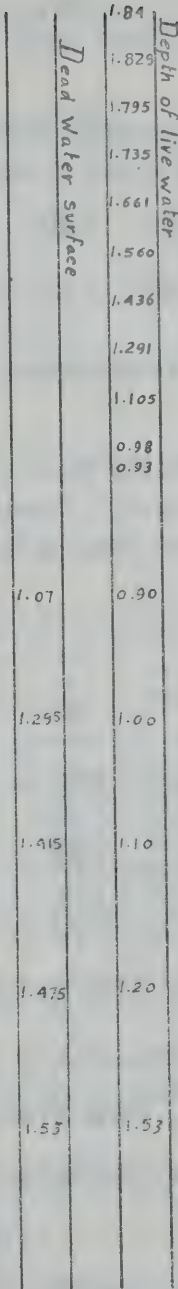
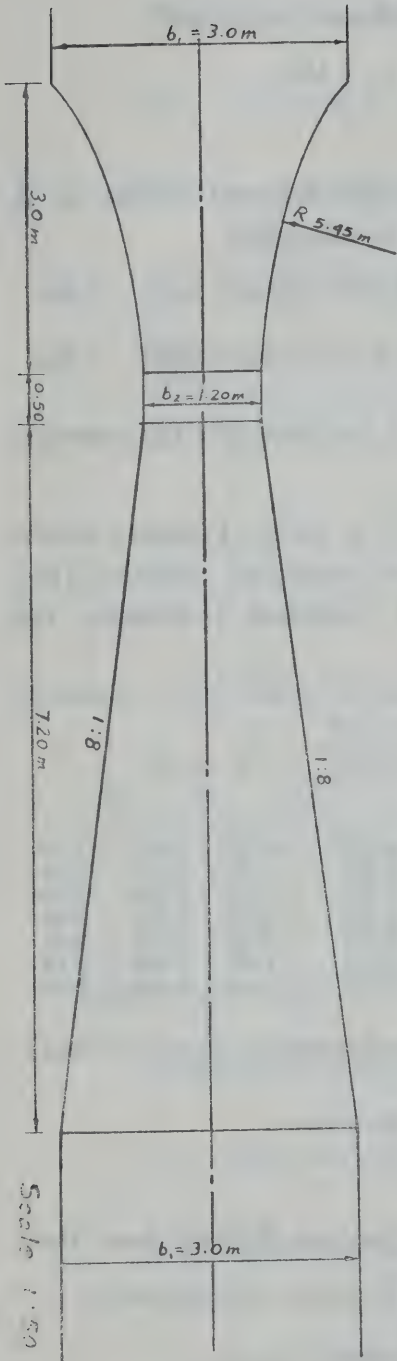


FIG. 23

Applying equations 22, 24, 25, 26, 27 and 29, we get :

$$h_1 = 1.84 \text{ m}, h_k = 1.38 \text{ m}, a = 0.105,$$

$$x = 0.20$$

Therefore applying equation (28) for different values of X (varying between 0 and 3.50 m), Z can be calculated.

X_m	0	0.50	1.00	1.50	2.00	2.50	3.00	3.50
Z_m	1.84	1.829	1.795	1.735	1.661	1.561	1.436	1.201

Determination of the water surface in the divergence (downstream the throat) :

Making use of the curves (Fig. 16) " λ " at different sections within the divergence are computed. Therefore, applying equation (21) the water surface at these sections are computed (neglecting the term ΔZe).

Section from 0-0 m.	b m.	h m.	V m/sec	$\frac{V_m^2}{2g}$	$\frac{V_m^2 - V_{m1}^2}{2g}$	$\bar{L}_v$	$\frac{v}{L_v}$	λ
3.5	1.20	1.291	3.61	0.665	0.058	0.50	0.069	3.20
4.0	1.325	1.105	3.77	0.723	0.055	1.00	0.139	2.26
4.5	1.45	0.98	3.90	0.778	0.082	1.50	0.208	1.46
5.0	1.575	0.86	4.10	0.860	0.110	2.00	0.278	1.00
5.5	1.70	0.75	4.35	0.970	0.130	2.50	0.347	0.72
6.0	1.825	0.657	4.65	1.10	0.124	3.00	0.415	0.62
6.5	1.95	0.58	4.90	1.224	0.116	3.50	0.488	0.52
7.0	2.075	0.52	5.13	1.34				

DETERMINATION OF THE JUMP

First Trial

Assume that the jump starts at distance 5.00 m. from 0-0, $h = 0.86 \text{ m}$, $b = 1.575 \text{ m}$:

$$F = 1.415, \text{ from (Fig. 21), } \frac{L}{h_{o2}} = 4.11$$

$$\therefore L = 6.30 \text{ m.}$$

Total energy head at beginning of jump “ H_{o1} ”

at $\frac{\bar{1}}{L} = 0.0, \beta = 1.25, \text{ and } \alpha = 1.00$

$$P_1 = 0.462 \text{ tons, } \left(\frac{P}{y} + z \right) = 0.970 \text{ m.}$$

$$\therefore H_{o1} = 0.970 + 0.860 = \underline{1.83 \text{ m.}}$$

Total energy head at end of jump “ H_{o2} ”

at $\frac{\bar{1}}{L} = 1.00, \beta = 1.0, \text{ and } \alpha = 1.42$

$$\left(\frac{P}{y} + z \right) = 1.53 \text{ m and } \alpha V_{o1}^2 / 2g = 0.0105 \text{ m.}$$

$$\therefore H_{o2} = 1.53 + 0.01 = 1.54 \text{ m.}$$

$$\therefore \Sigma \Delta Ze = 1.83 - 1.54 = \underline{0.29 \text{ m.}}$$

Strip No. 1: 1.50 m length, $\frac{\bar{1}}{L} = 0.23$

$$\beta_1 = 1.25, \alpha_1 = 1.00, \alpha'_1 = 1.02$$

$$\beta_2 = 1.125, \alpha_2 = 1.03, \alpha'_2 = 1.025$$

from equation (38) $\overline{\Delta Ze} = 0.087 \text{ m.}$

Applying the total energy head equation (35):

$$0.087 = 0.97 - D_2 + 0.229 \left(\frac{3.8 h_2^2 - 1.84}{h_2^2} \right)$$

For arbitrary values of h_2 , this equation is used to find out the corresponding values of D_2 .

$h_2 \text{ m}$	0.70	0.80	0.90	1.00	1.10
$D_2 \text{ m}$	0.895	1.105	1.236	1.332	1.408

Applying the impulse equation (30) :

$$P_1 = 0.726 \text{ tons}, P_2 = \left(D_2 h_2 - \frac{h_2^2}{2} \right) 1.95 \text{ tons (equ. 31)}$$

$$E_2 = (D_2 - 1.125 h_2) 1.46 \text{ tan is tons. (equ. 32)}$$

$$R = \left(0.462 + D_2 h_2 - \frac{h_2^2}{2} \right) 0.187 \text{ tons. (equ. 33)}$$

$$F' = 0.0152 (4.385 + h_2) \text{ tons. (equ. 34)}$$

Since $k_o = 1.63$ for throat ratio 0.40 and divergence
1 : 8 (Fig. 19).

$$M_2 - M_1 = 0.565 (2.93 / h_2 - 4.15) \text{ tons.}$$

Again, this equation is used to find out the values of D_2
corresponding to the selected values of h_2 .

$$h_2 \text{ m} \quad 0.70 \quad 0.80 \quad 0.90 \quad 1.00 \quad 1.10$$

$$D_2 \text{ m} \quad 0.92 \quad 1.10 \quad 1.23 \quad 1.32 \quad 1.38$$

Plotting the values of h_2 and D_2 in the two tables, we get :

$$h_2 = 0.077 \text{ m, and } D_2 = 1.043 \text{ m, } \therefore s = 0.173 \text{ m.}$$

$$\text{Strip No. 2 : 1.50 m. length, } \frac{\bar{I}}{L} = 0.475$$

$$\beta_1 = 1.125, \alpha_1 = 1.03, \alpha'_1 = 1.025$$

$$\beta_2 = 1.05, \alpha_2 = 1.07, \alpha'_2 = 1.03$$

from equation (38), $\overline{\Delta Z_e} = 0.174 \text{ m.}$

Applying the total energy head equation 35 :

$$0.087 = 1.043 - D_2 + 0.134 \left(\frac{5.4 h_2^2 - 2.25}{h_2^2} \right)$$

h_2 m	0.80	0.90	1.10	1.20
D_2 m	1.209	1.310	1.431	1.471

Applying the impulse equation 30 :

h_2 m	0.80	0.90	1.10	1.20
D_2 m	1.24	1.31	1.38	1.40

Plotting the values of h_2 and D_2 in the two tables, we get :

$$h_2 = 0.90 \text{ m} , \text{ and } D_2 = 1.31 \text{ m} , \therefore s = 0.36 \text{ m}.$$

Strip No. 3 : 1.50 m length, $\frac{\bar{I}}{L} = 0.714$

$$\beta_1 = 1.05 , \alpha_1 = 1.07 , \alpha'_1 = 1.03$$

$$\beta_2 = 1.01 , \alpha_2 = 1.21 , \alpha'_2 = 1.06$$

from equation (38), $\overline{\Delta Z_e} = 0.232 \text{ m}.$

$$\therefore \Delta Z_e \text{ for the 3rd strip} = \underline{0.058 \text{ m}}.$$

Applying the total energy head equation (35) :

h_2 m	0.90	1.00	1.10	1.20	1.30
D_2 m	1.356	1.415	1.457	1.491	1.517

Applying the impulse equation (30), we get :

h_2 m	0.90	1.00	1.10	1.20	1.30
D_2 m	1.370	1.415	1.445	1.465	1.485

Plotting the values of h_2 and D_2 in the two tables, we get :

$$h_2 = 1.00 \text{ m} , D_2 = 1.415 \text{ m} , \therefore s = 0.405 \text{ m}.$$

Strid No. 4: 1.20 m length, $\frac{\bar{I}}{L} = 0.905$

$$\beta_1 = 1.01, \alpha_1 = 1.21, \alpha'_1 = 1.06$$

$$\beta_2 = 1.00, \alpha_2 = 1.37, \alpha'_2 = 1.15$$

from equation (38), $\overline{\Delta Ze} = 0.273$ m.

$\therefore \Delta Ze$ for the 4th strip = 0.041 m.

Applying the total energy head equation 35:

$$0.041 = 1.415 - D_2 + 0.308 \left(\frac{9.0 h_2^2 - 7.30}{h_2^2} \right)$$

h_2 m	1.00	1.10	1.20	1.30	1.40
D_2 m	1.424	1.465	1.495	1.518	1.533

Applying the impulse equation (30), we get:

h_2 m	1.00	1.10	1.20	1.30	1.40
D_2 m	1.430	1.466	1.480	1.501	1.515

Plotting the values of h_2 and D_2 in the two tables, we get:

$$h_2 = 1.10 \text{ m, and } D_2 = 1.466 \text{ m, } \therefore s = 0.366 \text{ m.}$$

The point of intersection of the live water and dead water profiles does not coincide with the water surface on the downstream. The computations are repeated again assuming the beginning of the jump at another point within the divergence.

Secoond Trial: Assume that the jump starts at distance 4.70 m. from 0. The repeated calculations lead to the result that the point of intersection of the live water and dead water profiles, (Fig. 23), occurs at 11.40 m, from 0, and coincides with the downstream surface, *i.e.* at depth of 1.53 m.

$$\therefore L = 11.40 - 4.70 = 6.70 = \text{length of the jump.}$$

It is obvious that the solid floor must not end before the end of the jump.

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MINIMISING THE LOSS IN HEAD INSIDE THE 'HYDRAULIC JUMP' BY USING A "HUMP"

BY

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§ 1.—INTRODUCTION

It is known, that if a hydraulic jump is produced downstream a water structure, the major part of the energy loss takes place within the jump. That is why in the study, emphasise is given on the energy loss in the jump and on how this loss could be recovered.

In his work entitled "The energy loss in some irrigation works"⁽¹⁾ A. Kamel, has shown that the energy loss in the Venturi-flume could be reduced to about 45 %, if the bed beneath the jump is given an upward slope in the direction of flow forming a "hump". This reduction in the energy loss (about 55 %) which can be considered as a gain in the energy head (compared with the case of flume with horizontal bed), corresponds to a ratio of about 0.30 between the height of the hump " y " and the maximum water depth in the upstream " h_1 " (Fig. 1).

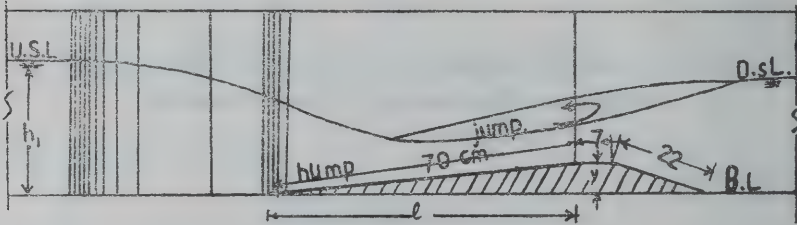
A similar gain in the energy head could be obtained in the case of the sluice gate (Fig. 2). In this case, the gain is 27 %.

A. Kamel, studied two irrigation works only, namely the Venturi-flume and the sluice gate; but since the problem of minimising the loss in head downstream canal outlets, is really a problem of a great importance in order to ensure "Free irrigation" (*i.e.* irrigation

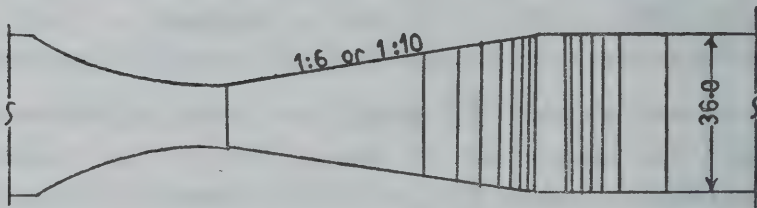
(1) M. Sc. thesis submitted to the Faculty of Engineering, Cairo University 1958.

without puming of water), the writer has found, that it will be certainly ussful if the effect of the hump in connection with another irrigation work, such as the standing wave weir could also be studied.

DIRECTION OF FLOW

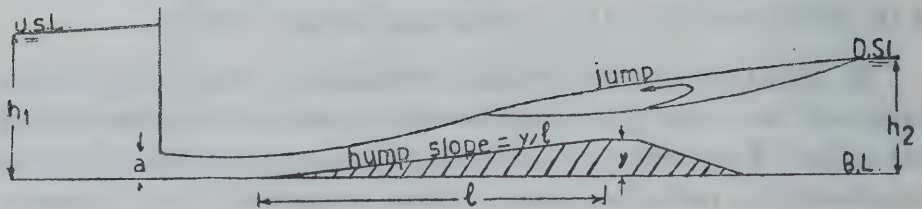


Long. sec. in a venturi flume with hump.



PLAN.

FIG. 1



Long. sec in a sluice gate with hump.

FIG. 2

§ 2.—CHOICE OF SHAPE OF STANDING WAVS WEIR USED
(Limits for the shape of the weir surface)

For the determination of the range of the slope, experiments were carried out on a model with a movable plate hinged to a broad crested weir at the crest level as shown in (Fig. 3). The model was fitted in a 30 cm. rectangular channel having two glass sides. The plate was set at different angles, with the horizontal, namely

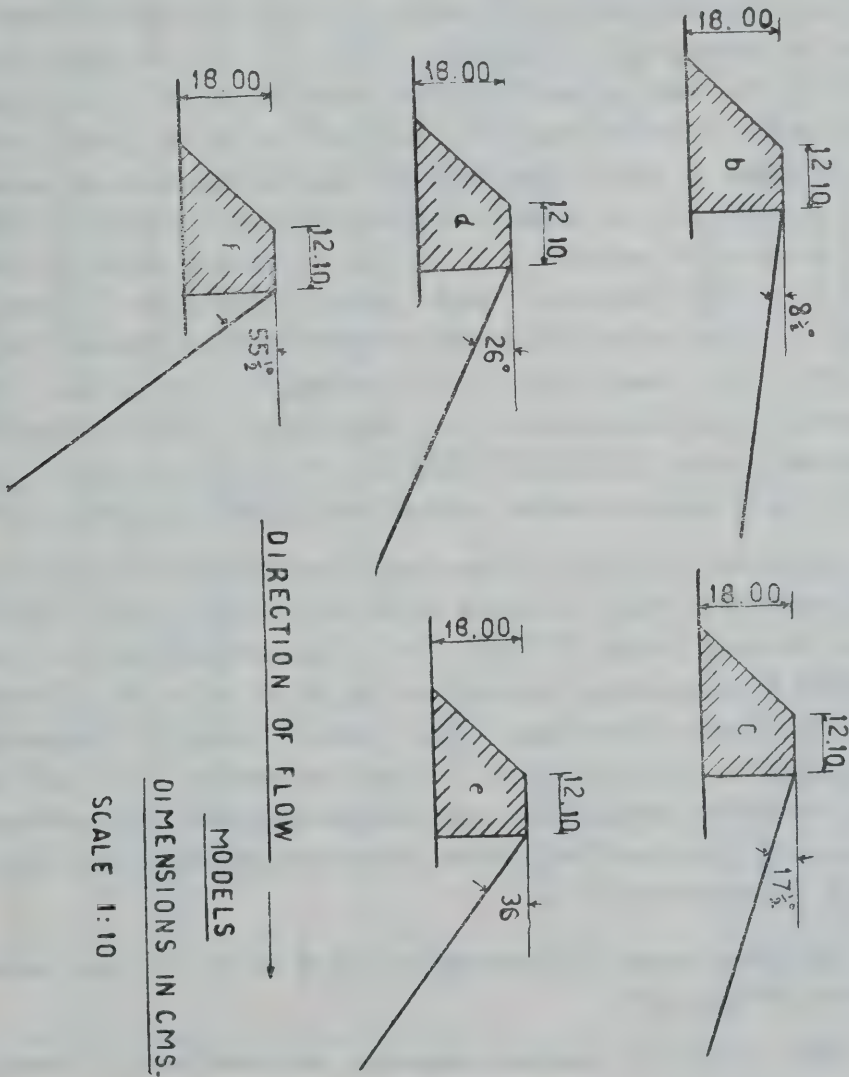


FIG 3

8.5°, 17.5°, 26°, 36° and 55.5°. For each position, tests were carried out at four different discharges. The water surfaces were measured and plotted. Pressures at different points on the bed for the different bed slopes were measured in the direction of the slope (Figs. 4, 5 and 6 for slopes 17.5°, 36° and 55.5°). The dotted lines represent the bed pressures. It is clear from (Figs. 4, 5 and 6) that these dotted lines have the tendency to drop down at a certain section a-a, which is practically the same for all cases. It is noticeable that this tendency increases with the inclination, *i.e.*, the bed pressure decreases with the increase of the inclination (Fig. 7). For an angle 55.5° (Fig. 6), the bed pressure becomes negative, and hence separation will result. Therefore, there is a particular inclination giving a zero bed pressure at section (a-a) (Section of minimum bed pressures). Such an inclination can be taken as the upper limit for the weir slope. Bed pressures at section (a-a) for the maximum discharge used (0.971 to 1.03 lit/cm/sec.) were plotted against the inclination (Fig. 7). The curve gives a zero pressure for an inclination angle of 42°. This means that if the inclination is more than 42°, separation at the bed at section (a-a) takes place. Therefore standing wave weirs with inclination more than 42° can be excluded. The inclination is therefore limited to values ranging between zero and 42°

Inclination less than 10° have not been considered because they will require very long structures which are not economical. Inclinations between 30 and 42° have also to be excluded because of the fact that for steep slopes, the overflowing jet cut under the tail-water with little surface disturbance. The hydraulic jump will thus act as submerged or drowned jump or even a submerged outlet, and the high velocity jet extends for a long distance along the floor and hence causes a big scour and requires an expensive long solid floor downstream the structure.

For these reasons, the reasonable slope of the weir surface ranges between 10° and 30°.

This study was made on a standing wave weir with an average surface slope of 20°.

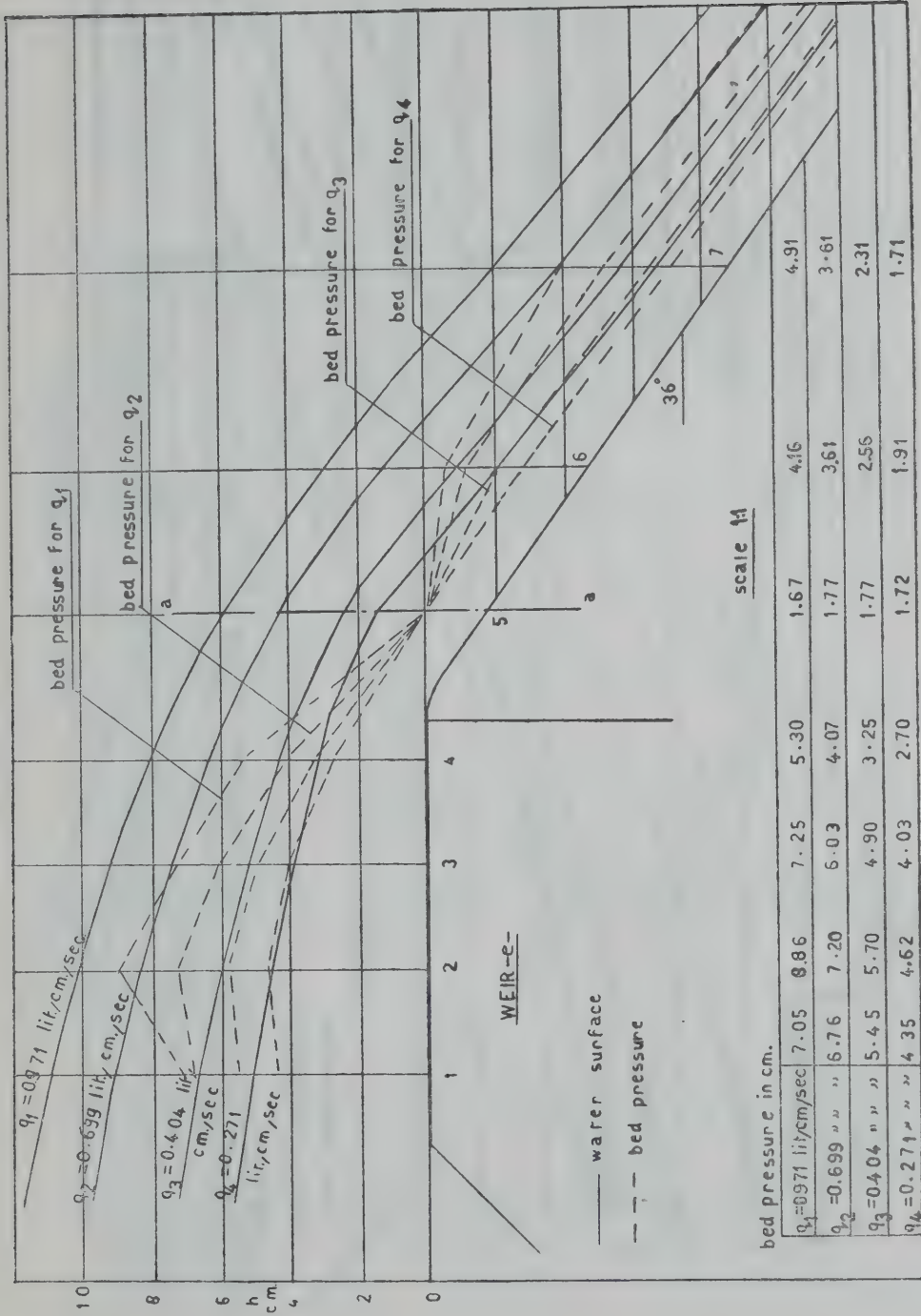


FIG. 4

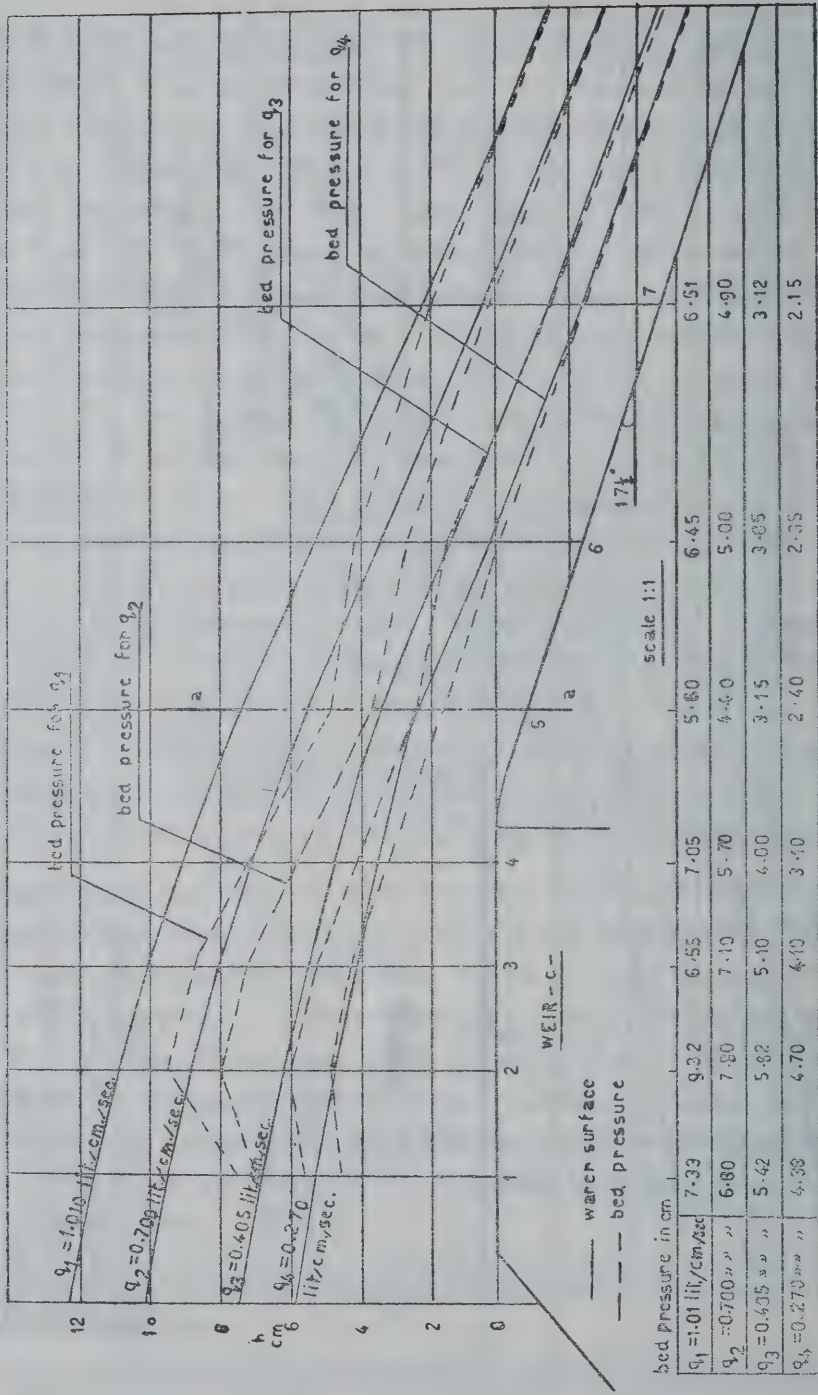


FIG. 5

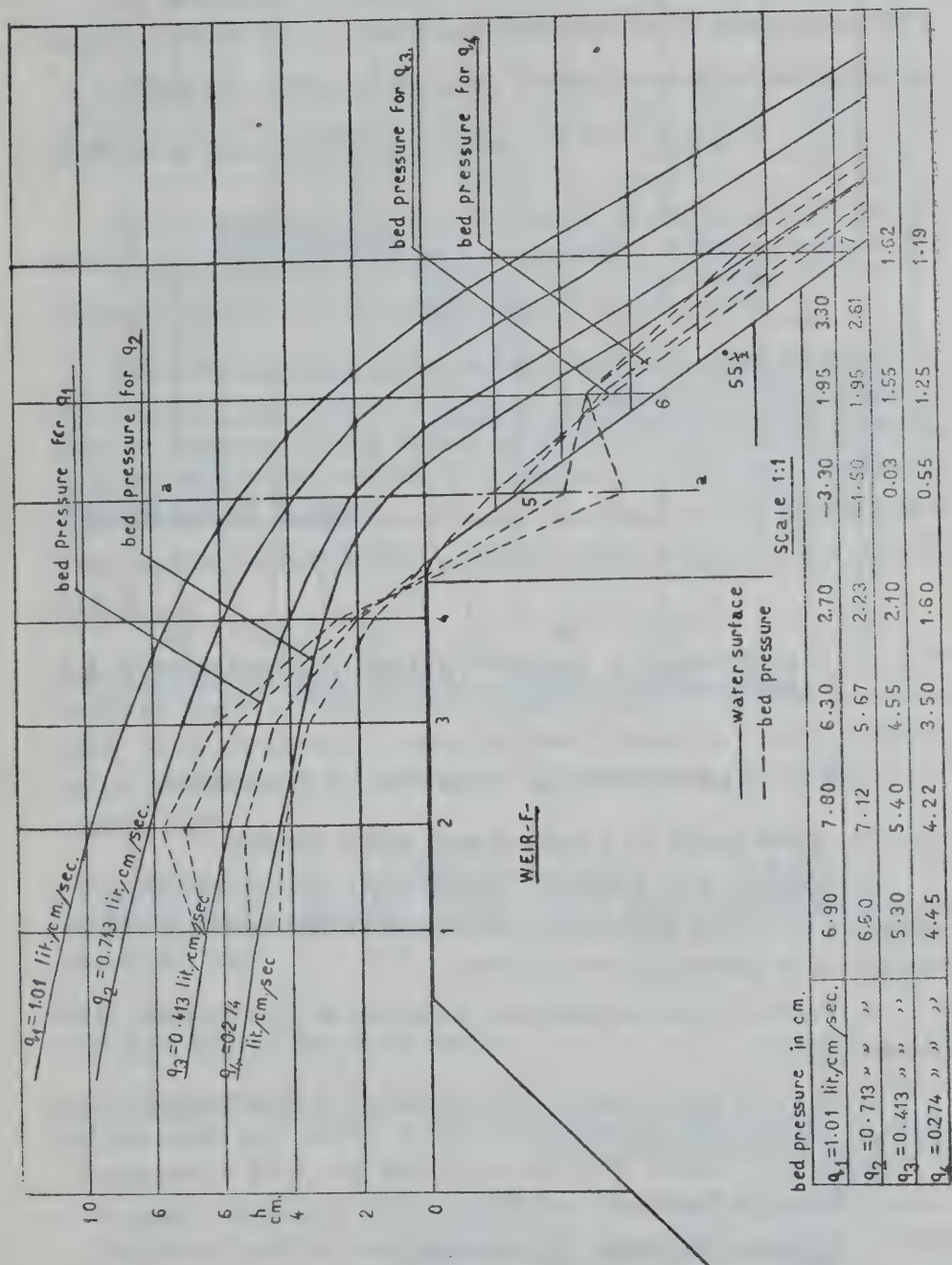


FIG. 6

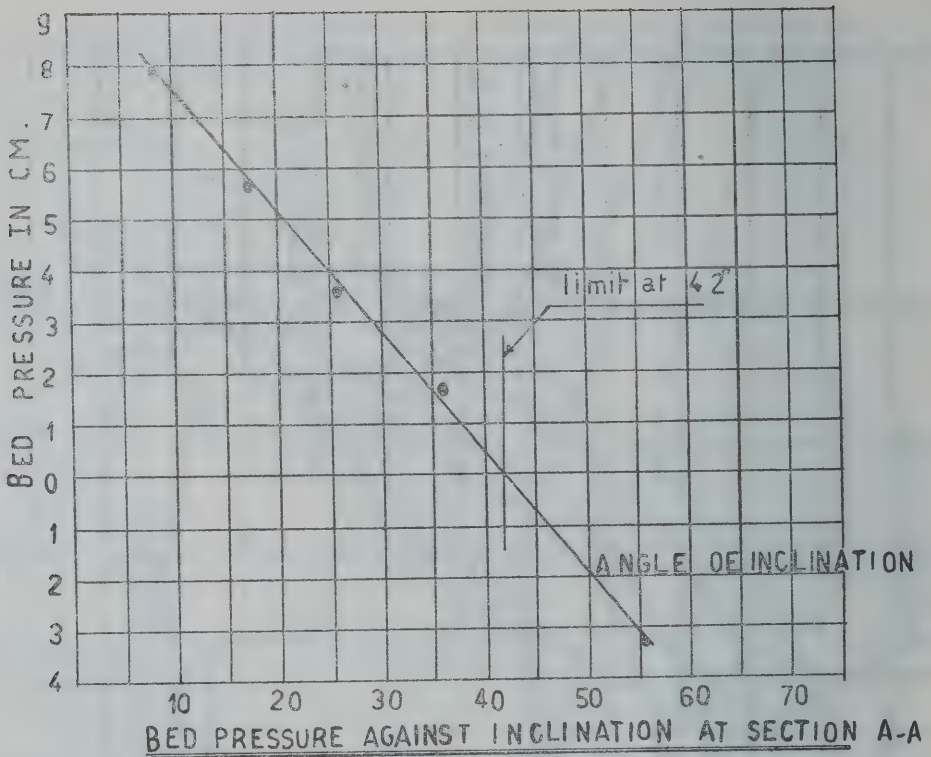


FIG. 7

§ 3.—THE THEORETICAL TREATMENT OF THE PROBLEM

(a) *The forces acting on a vertical strip within the jump :*

Considering any arbitrary vertical strip within the jump, 1-2 (Fig. 8), the forces acting upon the control surface enclosing the live water within that strip are as follows :

1. Friction forces between live water and solid boundaries of the channel, (F_2).
2. Shearing forces acting upon the stream surface demarking the live water from the roller, (F_1).
3. Pressure forces (P_1 and P_2).
4. Reaction of the bed upon the fluid :

(a) due to weight of the live and dead water, $\left(R_1 = \frac{W}{\cos i_n} \right)$.

(b) due to the dynamic pressures resulting from the centrifugal forces caused by the curvature and inclination of the stream lines, (R_o).

5. Reaction of the roller upon the fluid acting at the upper tide of the control surface enclosing the live water, $\left(R_2 = \frac{W}{\cos i_s} \right)$.

6. The corresponding rate of change of momentum within the control surface will be equivalent to the vector difference between the momenta entering and the control surface i. e. $\vec{M}_2 - \vec{M}_1$

Needless to say, all the acting forces are seriously influenced by the curvature and inclination of the stream lines. Hence, for an accurate estimate of the forces, these influencing factors has to be carefully taken into account. For instance, the in and out flowing momentum are obviously influenced by the direction of the stream lines, as well as the reactions exerted upon the live water, R_1 , R_o and R_2 .

Moreover, the frictional "shearing" forces between the live water on one hand and the solid boundaries on the dead water on the other hand, are affected as these are closely related to velocity changes, which are undoubtedly affected by the curvature and inclination of stream lines.

Therefore, it is justified to emphasise the influence of the curvature and inclination of the stream lines in formulating the equations used

(b) *The Energy loss in the jump :*

The energy loss in the jump which is lost partly in the rotation of the roller and partly in the formation of the eddies, is drawn continuously from the energy of the live water. The transfer of the energy from the "live water" to the "eddy dead water" takes place through the mechanism of momentum transfer creating shearing stresses between both layers. In other words the energy loss in the jump depends to a great extent on those created shearing

stresses, In order to minimise the energy loss in the jump a trial for reducing the shear stress will be necessary.

Since the amount of the shearing forces depend on the amount and direction of all forces acting on a vertical strip across the jump (Fig. 8) the evaluation of those forces and the condition of their equilibrium must be first determined.

I.—THE CONDITION OF EQUILIBRIUM OF THE FORCES ACTING ON A STRIP ACROSS THE JUMP

Applying the momentum theorem, the horizontal component of the forces acting upon the fluid within the control surface enclosing the live water for any given strip (Fig. 8), will be related by the following equations:

$$\left. \begin{aligned} (a) \quad P_1 - P_2 + E_1 \pm E_2 + E_c - F'_1 - F'_2 &= M_2 - M_1 \text{ (Fig. 8a)} \\ (b) \quad P_1 - P_2 \pm E_2 - F'_1 - F'_2 &= M_2 - M_1 \text{ (Fig. 8b)} \\ (c) \quad P_1 - P_2 - E_1 \pm E_2 - F'_1 - F'_2 - E_c &= M_2 - M_1 \text{ (Fig. 8c)} \end{aligned} \right\} \quad (1)$$

in which:

1.—*The pressure forces P_1 and P_2*

Taking $\left(\frac{P}{\gamma} + z \right)_m$ = the mean pressure head of the live water at any vertical section, s = the height of the “eddyding dead water” at the considered section excluding the air bubbles, then the pressure distribution diagram in the live water takes the shape shown in (Fig. 9).

$$\text{Then, } P = \left(\frac{P}{\gamma} \right)_m \cdot h + s \quad (2)$$

assuming that $\left(\frac{P}{\gamma} \right)_m$ occurs at $\frac{h}{2}$ from the bed, and putting

$$\left(\frac{P}{\gamma} + z \right)_m + s = D \quad (3)$$

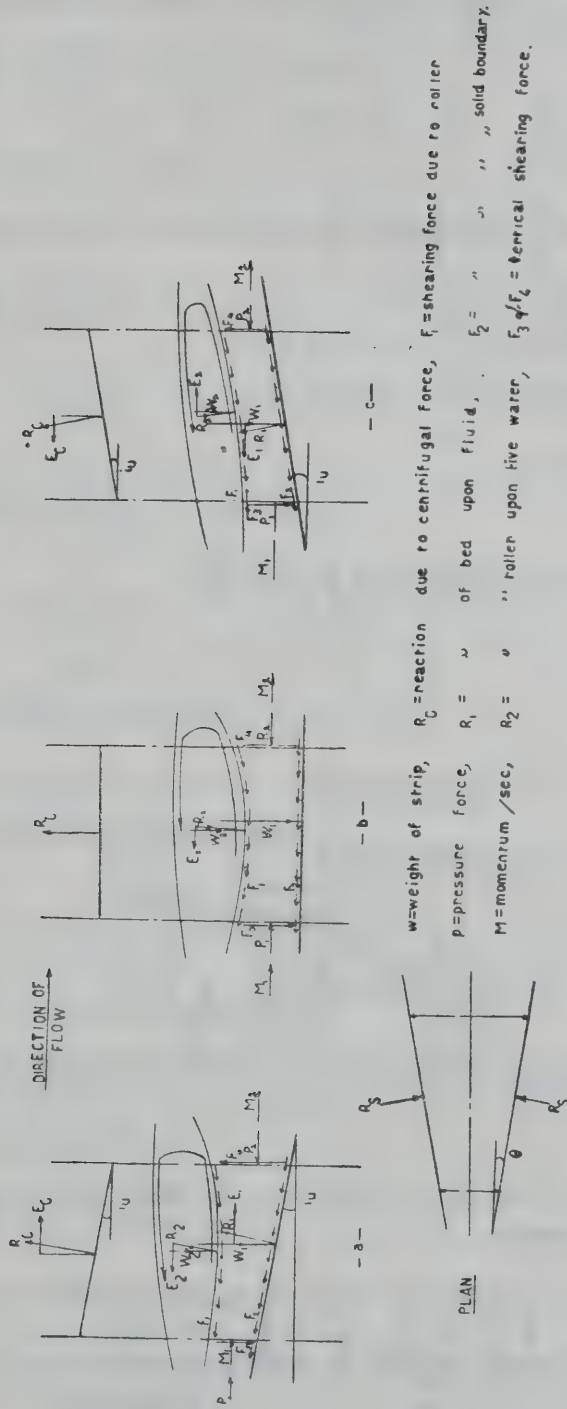


FIG. 8

Then $D = \left(\frac{P}{\gamma} \right)_m + \frac{h}{2} + s$

or $\left(\frac{P}{\gamma} \right)_m = D - \frac{h}{2} - s$

$\therefore P = \left(D - \frac{h}{2} - s \right) h + s h$

$= D h - \frac{h^2}{2} - s h + s h$

or $P = D h - \frac{h^2}{2} \quad \dots \dots \dots (4)$

2.—The horizontal components E_1 and E_2 :

$E_1 = \gamma \frac{(h_1 + s_1) + (h_2 + s_2)}{2} \cdot b \cdot l \cdot \tan i_u \quad \dots \dots (5)$

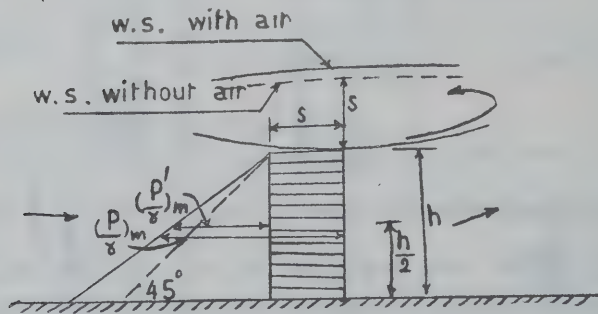


FIG. 9

Taking " β " defined as the ratio of the actual pressure to the hydrostatic pressure,

them $\beta = \frac{\left(\frac{P}{\gamma} + z \right)_m \cdot \frac{h}{2}}{\frac{h^2}{2}} = \frac{\left(\frac{P}{\gamma} + z \right)_m}{h}$

Substituting this value in equation (3) we get,

$$D = \left(\frac{P}{\gamma} + z \right)_m + s = \beta h + s$$

or $s = D - \beta h$ (6)

Substituting this value of "s" in equation (5) for "E₁"

we get
$$E_1 = \frac{\gamma}{2} [(h_1 + D_1 - \beta_1 h_1) + (h_2 + D_2 - \beta_2 h_2)] \text{ b. l. } \tan i_u$$

$$= \frac{\gamma}{2} [\{D_1 - (\beta_1 - 1)h_1\} + \{D_2 - (\beta_2 - 1)h_2\}] \text{ b. l. } \tan i_u \quad (7)$$

and
$$E_2 = \gamma \frac{s_1 + s_2}{2} \cdot \text{b. l. } \tan i_s$$

$$= \frac{\gamma}{2} [(D_1 - \beta_1 h_1) + (D_2 - \beta_2 h_2)] \text{ b. l. } \tan i_s \quad (8)$$

3.—The horizontal components F'_1 and F'_2 :

The horizontal component of the shearing force (between the live stream and the roller) is:

$$F_1 = \tau_s \cdot \frac{l \cdot b}{\cos i_s} \cdot \cos i_s = \tau_s \cdot l \cdot b.$$

where τ_s is the shear stress.

The horizontal component of the frictional force at the bed and sides of the channel F'_2 is:

$$F'_2 = \tau_o \cdot A \cos i_u$$

where τ_o = shear stress at the solid boundary,

A = the area of contact of the live water with the solid boundaries.

Putting $\tau_o = \frac{1}{2} \cdot \rho_o \cdot v_m^{-2} \cdot f_b$,

and using the Chezy's formula $C = \sqrt{\frac{2g}{f_b}}$

where $C = \text{Chezy's coefficient} \cong 60$ (in our case)

Then $60 = \sqrt{\frac{2g}{f_b}}$ or $f_b = 0.0054$

4.—The Momenta M_1 and M_2 :

The momentum $m v_m = \frac{W}{g} v_m$ will be multiplied by a coefficient α' where:

$$\alpha' = \int_0^A \frac{(v dA) X V}{(A v_m) V_m} \text{ where:}$$

v = is the horizontal component of the actual velocity V .

$$\text{i.e. } V = v \sec i_m$$

5.—The horizontal force E_c :

It is a known fact that with curved stream lines the pressure distribution will deviate from the usual hydrostatic one. With concave stream lines as for case of the live water within the jump, the total pressure along the bed will be higher than its hydrostatic value, (Fig. 10). This force can be estimated with the help of the pressure distribution diagrams obtained for traverses 1-1 and 2-2.

$$\therefore \left(\frac{\Delta P}{\gamma} \right)_{c_{1-2}} = \frac{1}{2} \left[\left(\frac{\Delta P}{\gamma} \right)_{c_1} + \left(\frac{\Delta P}{\gamma} \right)_{c_2} \right],$$

The total dynamic force acting upon the fluid can be put as:

$$R_c = \frac{\gamma}{2} \left[\left(\frac{\Delta P}{\gamma} \right)_{c_1} + \left(\frac{\Delta P}{\gamma} \right)_{c_2} \right] b \cdot \frac{1}{\cos i_u}$$

∴ The horizontal component of this force, E_c , which counts for the effect of the curvature and inclination of the stream lines is :

$$E_c = \frac{\gamma}{2} \left[\left(\frac{\Delta P}{\gamma} \right)_{c_1} + \left(\frac{\Delta P}{\gamma} \right)_{c_2} \right] b \cdot \frac{1}{\cos i_u} \cdot \sin i_u \quad (9)$$

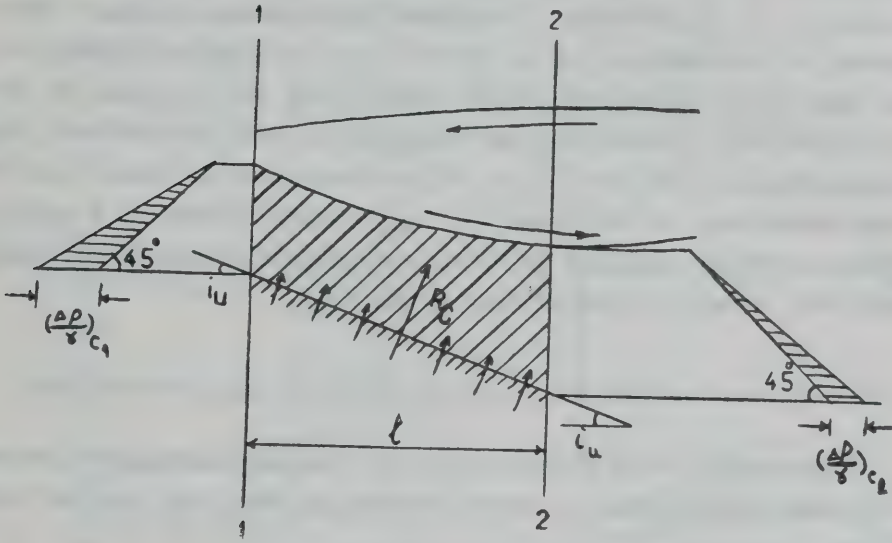


FIG. 10

The shearing forces at the plane between the live stream and the eddying dead water, (the roller) can be obtained from equation (1) as follows :

$$\left. \begin{aligned} (a) \quad F'_1 &= (M_1 + P_1 \pm E_2 + E_c + E_1) - (M_2 + P_2 + F'_2) \\ (b) \quad F'_1 &= (M_1 + P_1 \pm E_2) - (M_2 + P_2 + F'_2) \\ (c) \quad F'_1 &= (M_1 + P_1 \pm E_2 - E_c - E_1) - (M_2 + P_2 + F'_2) \end{aligned} \right\} \quad (1)'$$

Comparing the values of F'_1 computed from the above three equations, we find that the value obtained from equation (c) that corresponds to the case of the hump has the least value.

This means that, if the bed under the jump is given an upward slope in the direction of the flow forming a hump, the loss in head in the jump, is greatly reduced.

Shear stresses were calculated once for the channel with horizontal bed and another for the channel with a hump slope 7 : 100, according to equation (1)'. The results are tabulated in Table (1) for a rate of discharge of 35.00 lit/sec. Comparing the values of these shear stresses for both cases, it can be noticed, that the shearing stresses for the case of the hump are smaller than those for the case of the horizontal bed. Calculating the energy losses in both cases from the simple equation $\Sigma \Delta Ze = H_1 - H_2$, we found that this small loss in head corresponds to the small shearing stresses, (Tables 2, 3 and 4) which proves the fact that by using a hump downstream the standing wave weir, minimized loss of head is obtained.

II.—EFFECT OF THE HUMP ON THE PATH OF THE ENERGY LINE

The energy line is defined as the line passing through the points of the total energy head in successive sections along the weir and the channel.

Formulating this definition mathematically for two successive sections (Fig. 11), we get :

The total energy head at section I,

$$H_1 = \left(\frac{P}{\gamma} + z \right)_{m1} + s_1 + \alpha_1 \frac{v_{m1}^2}{2g} \sec^2 i_{m1}, (\text{for horizontal bed})$$

$$H_1 = \left(\frac{P}{\gamma} + z \right)_{m1} + s_1 + y_1 + \alpha_1 \frac{v_{m1}^2}{2g} \sec^2 i_{m1} (\text{for the hump})$$

and the total energy head at section II :

$$H_2 = \left(\frac{P}{\gamma} + z \right)_{m2} + s_2 + \alpha_2 \frac{v_{m2}^2}{2g} \sec^2 i_{m2} (\text{for horizontal bed})$$

$$H_2 = \left(\frac{P}{\gamma} + z \right)_{m_2} + s_2 + y_2 + \alpha_2 \frac{v_{m_2}^2}{2g} \sec^2 i_{m_2} \text{ (for the hump)}$$

where y = the height of the sloping bed above the horizontal bed
at the considered section,

s = the height of the rolling dead water (Fi. 9).

α = kinetic energy factor

$$= \frac{2g}{V_m^3 A} \int_0^A \left(\frac{V^2}{2g} + \frac{P}{\gamma} + z \right) v \, dA$$

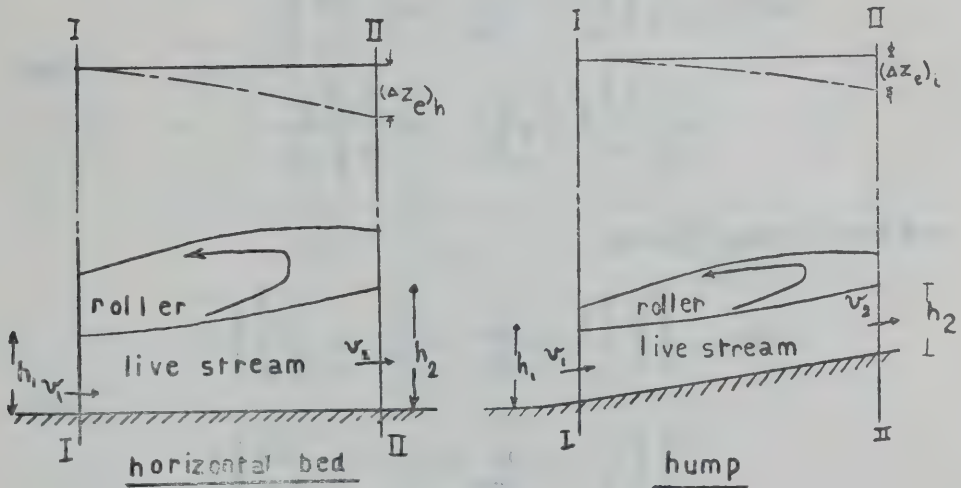


FIG. 11

i = slope of the tangent to any stream line.

As $V_m = v_m \sec i_m$, the above equation will be:

$$H_1 = \left(\frac{P}{\gamma} + z \right)_{m_1} + \alpha_1 \frac{V_{m_1}^2}{2g} \text{ (for horizontal bed)}$$

$$H_1 = \left(\frac{P}{\gamma} + z \right)_{m_1} + y_1 + \alpha_1 \frac{V_{m_1}^2}{2g} \text{ (for the hump)}$$

$$H_2 = \left(\frac{P}{\gamma} + z \right)_{m_2} + \alpha_2 \frac{V_{m_2}^2}{2g} \text{ (for horizontal bed)}$$

$$H_2 = \left(\frac{P}{\gamma} + z \right)_{m_2} + y_2 + \alpha_2 \frac{V_{m_2}^2}{2g} \text{ (for the hump)}$$

denoting the head loss along the strip as ΔZ_e

$$\therefore \Delta Z_e = H_1 - H_2$$

Therefore, the case of horizontal bed,

$$(\Delta Z_e)_h = \left[\left(\frac{P_1}{\gamma} + z \right)_{m_1} + \alpha_1 \frac{V_{m_1}^2}{2g} \right] - \left[\left(\frac{P_2}{\gamma} + z \right)_{m_2} + \alpha_2 \frac{V_{m_2}^2}{2g} \right] \quad (10)$$

and for the case of hump,

$$(\Delta Z_e)_i = \left[\left(\frac{P_1}{\gamma} + z \right)_{m_1} + y_1 + \alpha_1 \frac{V_{m_1}^2}{2g} \right] - \left[\left(\frac{P}{\gamma} + z \right)_{m_2} + y_2 + \alpha_2 \frac{V_{m_2}^2}{2g} \right] \quad (11)$$

Applying the above-mentioned total head equations on successive sections along the jump, the energy line can be determined.

Calculations were made, using these equations, for different discharges.

Fig. 12 shows the water surface profiles and the total energy lines for $Q = 35.00$ lit./sec. for horizontal bed and for the hump.

Comparing each pair of the energy lines (horizontal bed and hump), one can notice, that the energy line corresponding to the bed with hump is flatter than that with horizontal bed. (Fig. 12).

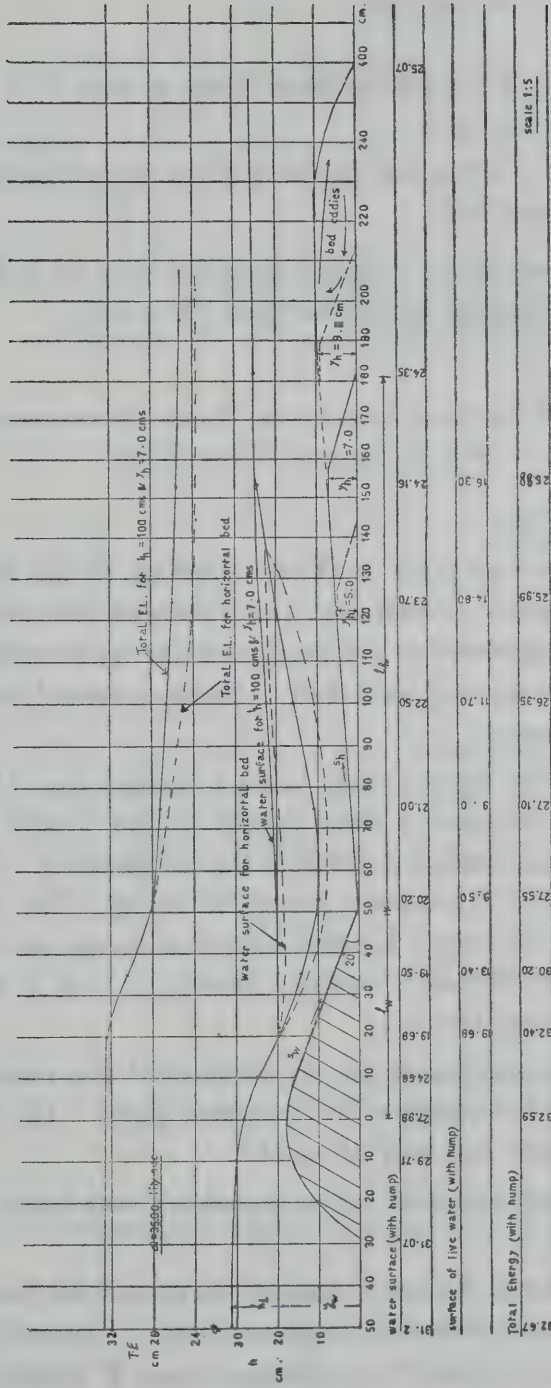


Fig. 12

$$i. e. \quad \frac{(\Delta Ze)_i}{l} \text{ is less than } \frac{(\Delta Ze)_h}{l}$$

where $(\Delta Ze)_i$ is the loss in head along a strip "l" for the bed with hump and

$(\Delta Ze)_h$ is the loss in head along the same strip for the horizontal bed.

It is also clear from (Tables 2, 3 and 4) that $(\Sigma \Delta Ze)_i$, is less than that for the case of horizontal floor, $(\Sigma \Delta Ze)_h$.

§ 4.—THE MOST EFFECTIVE HUMP DOWNSTREAM THE STANDING WAVE WEIR

A.—*Applied Study:*

The treatment made by A. Kamel leading to the design of the most effective hump downstream the Venturi-finne and the Sluice gate, cannot be applied for the design of the most effective Hump downstream the standing wave weir. Hence a special study for the latter case is necessary.

Therefore, five figures were plotted, namely figs. 12, 13, 14, 15 and. 16 Fig. 13 consists of three sets of curves giving the relation between the loss in head $\Sigma \Delta Ze$ and the discharge Q . Each set of curves corresponds to a certain length of hump. The lengths used were 70, 100 and 130 cms. By each length of hump, the experiments were carried for different heights, namely $y = 4, 7$ and 9 cms. (Figs. 13 a, 13 b and 13 c.)

A better representation can be obtained if the relation $(\Sigma \Delta Ze$ and $Q)$ is plotted for humps with a constant slope. The chosen constant slope of hump was 0.07 (Fig. 14).

Comparing the obtained curves shown on that figure, it is to be noticed that :

(a) In all cases, the loss in head in the case of the hump is always less than that in the case of the horizontal bed. This can be explained by comparing the values of the shearing force F'_1 obtained from the

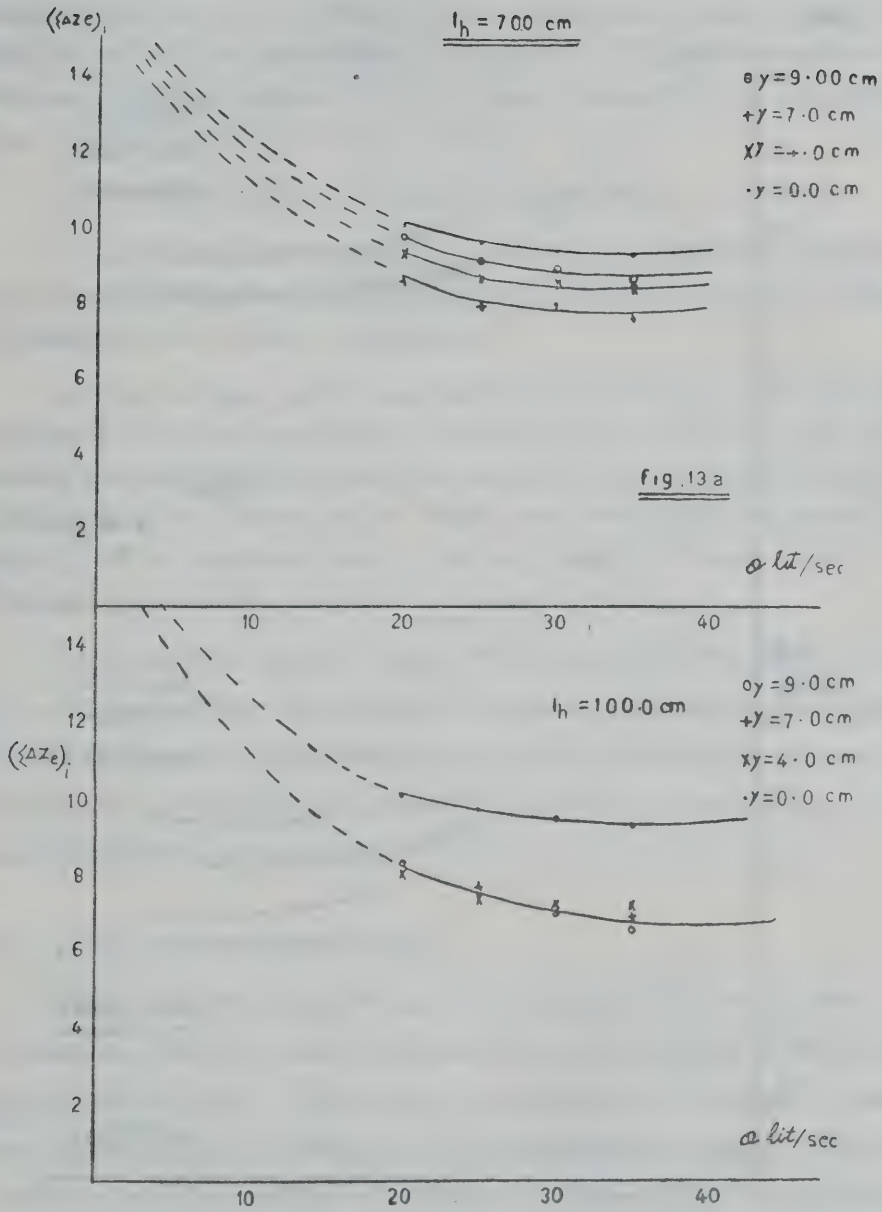


FIG. 13 a, b

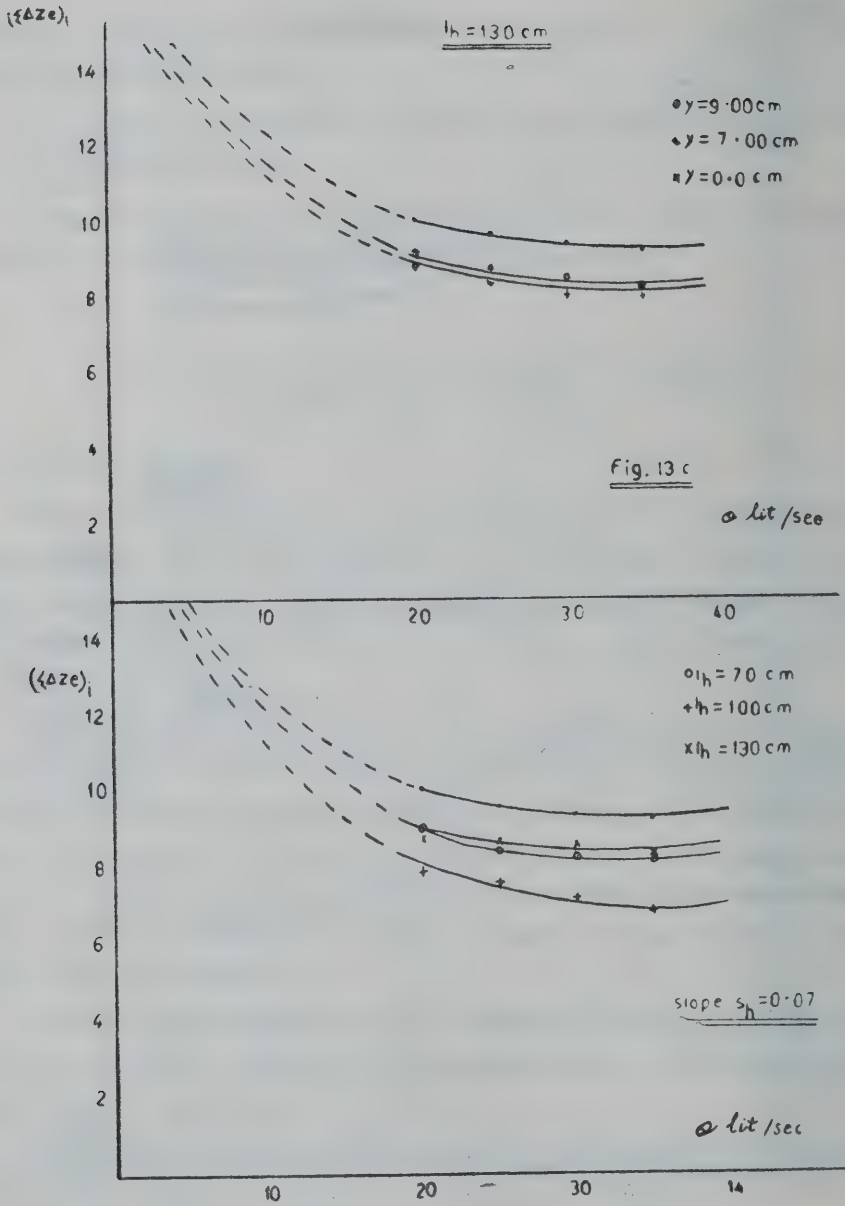


FIG. 13c and FIG. 14

group of equations (1)' with each other. One finds that the value obtained from equation (c) which corresponds to the case of the hump has the least value. Calculating the corresponding shear stresses and the loss in head we found that the small loss in head corresponds to the small shearing stresses (Table 1) which realises the idea of minimising the loss in head by using a hump.

(b) The most effective hump is the one with $l_h = 100$ cm.

In case of the short length ($l_h = 70$ cm), the hydraulic jump does not be completely over the hump (Fig. 12) and hence the loss in head in this case will not be the minimum.

In case of the long hump, however, where $l_h = 130$ cm, the hydraulic jump lies completely over the hump and the loss in head *within* the jump will be small, but the total loss in head $\Sigma \Delta Z_e$ which includes the loss due to the bed eddies produced just downstream the hump will be increased due to the big height of hump, $y = 9$ cm. Hence $\Sigma \Delta Z_e$ in this case also will not be minimum.

The minimum loss in head corresponds to the moderate length ($l_h = 100$ cm.) where the hydraulic jump lies completely over the hump and in the same time the losses due to the bed eddies downstream the hump are not so effective because y in this case is smaller ($y = 7.00$ cm) Fig. 12.

B.—Design of the effective Hump :

Since the hump is a means of minimising the loss in head, its dimensions and slope must be dependent on the Energy of the falling water over the weir. This energy is a function of the height of water over the crest of the weir h_1 and the downstream slope of the weir surface s_w . The relation which gives the dimensions of the hump must therefore contain these valuable factors. The values of $\frac{(\Sigma \Delta Z_e)_i}{(\Sigma \Delta Z_e)_h}$ and $\left(\frac{s_h}{s_w} \times \frac{y_h}{h_1} \right)$ for all the experimented plates and slopes

In § 2, it was shown that the convenient inclination of the weir surface ranges between 10° and 30° . This study was made for a standing wave weir with an average surface inclination of 20° . Putting in equation (12), $\tan 20^\circ$ instead of s_w , we get :

$$\frac{s_h}{\tan 20} \times \frac{y_h}{h_1} = 0.22$$

$$\text{or } \frac{y_h}{l_h} \times \frac{y_h}{\tan 20 \times h_1} = 0.22$$

$$\therefore y_h^2 = 0.22 \tan 20 \times h_1 \times l_h \quad . \quad . \quad . \quad (13)$$

Putting in this equation for h_1 , the value $(h_1)_a$ corresponding to the average discharge Q_a where $Q_a = \frac{Q_{\max.} \times Q_{\min.}}{2}$, we get $y_h^2 = 0.22 \tan 20 \times (h_1)_a \times l_h \quad . \quad . \quad . \quad (14)$

This last equation still contains two unknowns namely y_h and l_h the main dimensions of the hump. To solve this equation, a second relation must therefore be given.

values of $\frac{(\sum \Delta Z e)_i}{(\sum \Delta Z e)_h}$ against values of $\frac{l_h}{l_w}$ are plotted in Fig. 16.

This curve also has two limits :

(a) For $\left(\frac{l_h}{l_w}\right) = \text{zero } i.e. \text{ for } h = \text{zero (case of horizontal bed)}$, $\frac{(\sum \Delta Z e)_i}{(\sum \Delta Z e)_h} = \frac{(\sum \Delta Z e)_h}{(\sum \Delta Z e)_h} = \text{one}.$

(b) For $\left(\frac{l_h}{l_w}\right) = \text{infinity, } i.e., \text{ for a very long hump with respect to length of weir } l_w$, $\frac{(\sum \Delta Z e)_i}{(\sum \Delta Z e)_h} = \text{one}.$

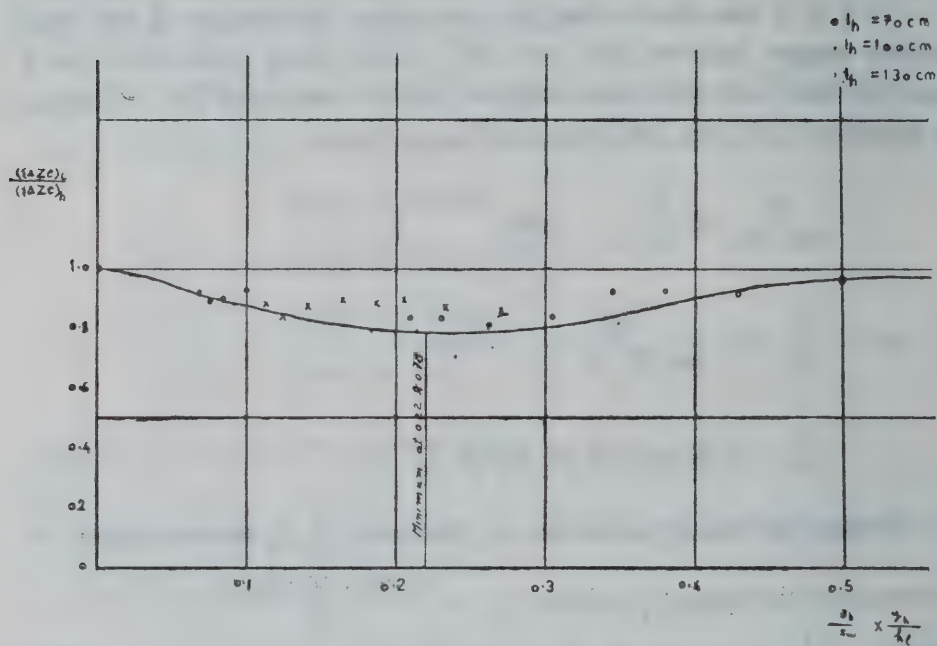


FIG. 15

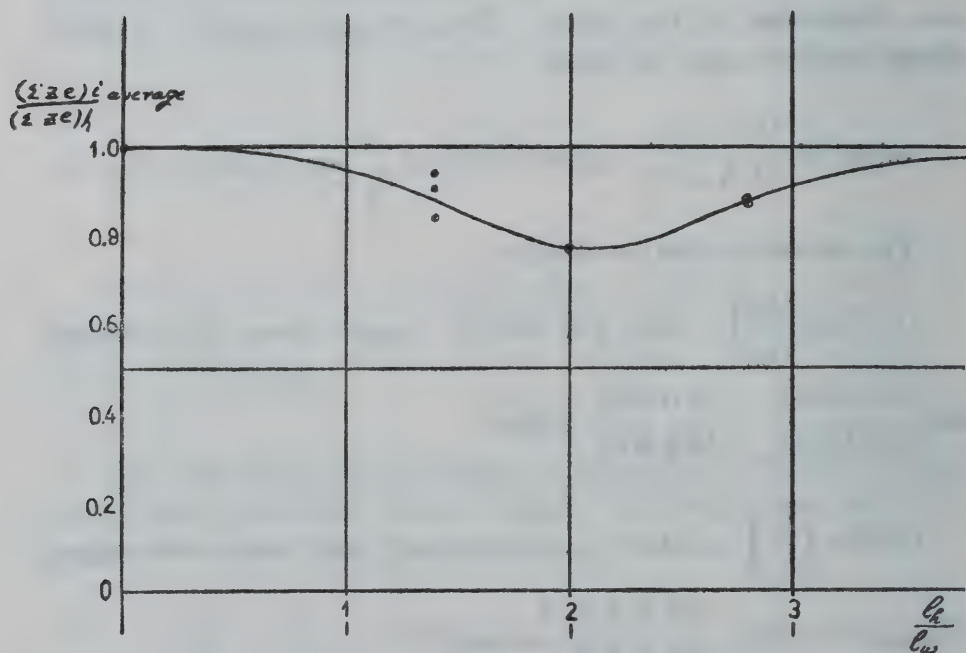


FIG. 16

$$\therefore y_w = 2.50 \text{ m.}$$

$$l_w = 7.00 \text{ m.}$$

$\therefore$ from equation (15)

$$\begin{aligned} y_h^2 &= 0.44 y_w (h_1)_a \\ &= 0.44 \times 2.50 \times 1.0 = 1.1 \end{aligned}$$

$$\therefore y_h = \underline{1.05 \text{ m.}}$$

and from equation (16)

$$l_h = \frac{2 y_w}{\tan 20} = 2 \times 7.0 = \underline{14.0 \text{ m.}}$$

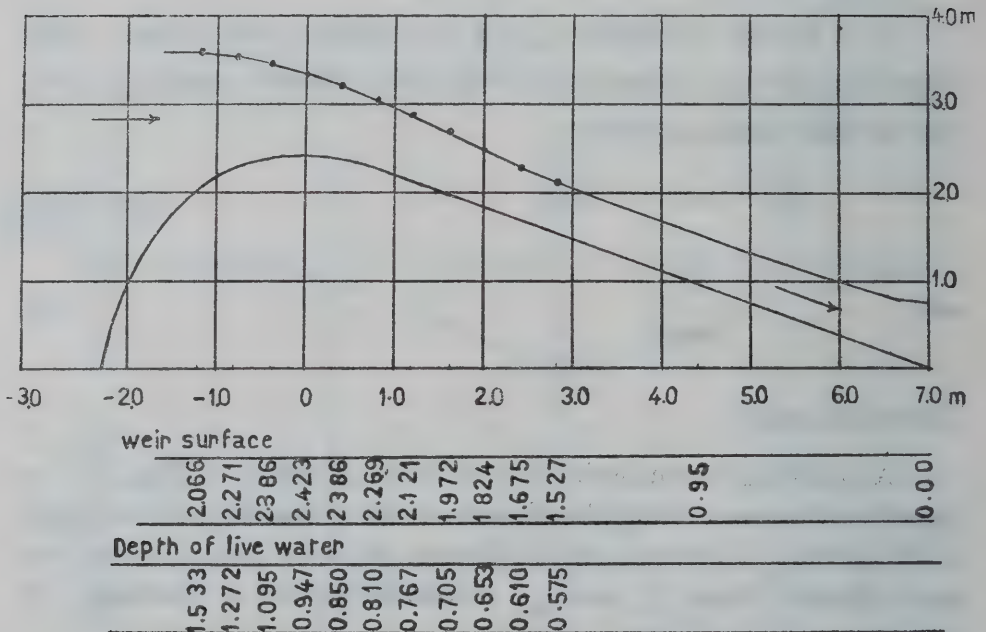


FIG. 17

N.B.: Length of hydraulic jump for the case of horizontal bed = 13.00 m. (measured from the toe of the weir).

TABLE I

With horizontal bed

Computations of shear stresses

Strip bet. Sections	V ₀₁ cm./sec	P _j Kgs.	P ₂ Kgs.	M ₁ Kgs.	M ₂ Kgs.	E ₁ Kgs.	E ₂ Kgs.	E _c Kgs.	F ₂ ¹ Kgs.	F ₁ ¹ Kgs.	τ, gm./cm. ²
20 and 40	159.0	0.93	2.90	5.695	5.01	2.87	0.928	0.456	0.058	1.055	1.460
40 and 65	159.0	2.90	4.49	5.01	4.30	1.96	0.558	0.272	0.061	0.733	0.820
65 and 90	159.0	4.49	5.51	4.30	3.55	0	-0.675	0	0.041	0.364	0.405
90 and 120	159.0	5.51	7.46	3.55	2.63	0	-1.300	0	0.032	0.238	0.224

With Hump, l_h = 100 cm., y_h = 7.0 cm.

20 and 35	159.0	0.93	2.65	5.695	4.80	2.05	0.62	0.184	0.044	0.745	1.38
35 and 53	159.0	2.65	5.39	4.80	3.79	2.79	1.22	0.708	0.022	0.526	0.80
53 and 75	159.0	5.39	4.75	5.79	4.06	-1.06	-1.05	-0.160	0.018	0.202	0.255
75 and 100	159.0	4.75	5.61	4.06	3.15	-1.195	-1.35	-0.025	0.029	0.181	0.200
100 and 125	159.0	5.61	5.85	3.15	2.45	-1.18	-0.905	-0.009	0.0169	0.159	0.176
125 and 153	159.0	5.85	4.77	2.45	2.40	-1.24	-0.274	0	0.0205	0.140	0.139

TABLE 4

$l_h = 130$ cm.

Section—50 cm., U. S. Weir.

Q lite/sec.	t' cm.	h cm.	$\left(\frac{P}{y} + z\right)_m$	$\alpha \frac{V_m^2}{2g}$	T. E.	h_1 on weir cm.	$\frac{l_h}{l_w}$
35.0	0.90	31.28	32.18	0.49	32.67	9.12	2.60
30.0	0.90	30.16	31.06	0.39	31.45	8.30	2.60
25.0	0.90	28.70	29.60	0.30	29.90	7.31	2.60
20.0	0.90	27.20	28.10	0.22	28.32	6.28	2.60

Section 400 cm., D. S. Weir

Q lit/sec	t' cm.	h cm.	$\left(\frac{P}{y} + z\right)_m$	$\alpha \frac{V_m^2}{2g}$	T. E.	$(\Sigma \Delta ze)_i$	$\frac{(\Sigma \Delta ze)_i}{(\Sigma \Delta ze)_h}$	$\frac{s_h}{s_w} \times \frac{y_h}{h_1}$
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$y_h = 9.00$ cm., $S_h = 0.069$

35 0	0.38	23.15	23.53	0.91	24.44	8.23	0.895	0.187
30.0	0.38	21.70	22.08	0.77	22.85	8.6	0.905	0.205
25.0	0.38	20.35	20.73	0.56	21.29	8.51	0.88	0.234
20.0	3.38	18.69	19.07	0.43	19.50	8.82	0.875	0.271
							0.89	

$y_h = 7.00$ cm., $S_h = 0.054$

35.0	0.38	23.25	23.63	0.89	24.52	8.15	0.885	0.113
30.0	0.38	22.38	22.76	0.68	23.44	8.01	0.845	0.124
25.0	0.38	20.35	20.73	0.56	21.29	8.61	0.89	0.141
20.0	0.38	18.47	18.85	0.46	19.31	9.01	0.915	0.164
							0.88	

Horizontal bed. (No Hump) $(\Sigma \Delta ze)_h$

35.0	0.38	22.15	22.53	0.93	23.46	9.21	1.00	
30.0	0.38	20.83	21.21	0.75	21.96	9.49	1.00	
25.0	0.38	19.20	19.58	0.62	20.20	9.70	1.00	
20.0	0.38	17.38	17.79	0.48	18.24	10.08	1.00	

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MODIFIED SUTRO-WEIR INVESTIGATIONS

BY

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I.—EXPERIMENTAL DESIGN

SYNOPSIS

The proportional flow notch or Sutro-weir has that particular shape which gives a linear head discharge relationship. This straight line flow makes it useful for grit-chamber design and for float controlled dosing devices where it is possible to maintain automatically a nearly constant velocity with quite large variations in flow. As it is practically impossible to make the weir crest of infinite length, many investigators tried some approximate method to limit this length. In this investigation a similar trial is experimented with and the results obtained are exhibited and discussed.

HISTORICAL REVIEW

According to B. D. Moses⁽²⁾ in 1915, Sutro was the first to propose the use of weirs giving discharges proportional to the heads over the weir. The outline of such weirs was developed graphically and not analytically. The general shape is such that the theoretical discharge is either directly proportional to the head or a linear function of the head, provided that the head does not fall below a certain limit.

E. W. Rettger⁽³⁾ in (1914), realised the fact that a weir of such shape would offer several advantages. He therefore proposed to develop analytically the shape of a weir for which the theoretical discharge is directly proportional to the head. He made a theoretical

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approach without any supporting experimental data. He found that a weir and whose width at any point is inversely proportional to the square root of the height of that point above the crest could have a theoretical discharge exactly proportional to the head Fig. 1 a.

E. A. Pratt⁽⁴⁾ in (1914), was given the task of developing the theory of the weir used by Sutro. In this study, instead of employing the simple form in which the breadth of the weir approaches infinity as the head approaches zero, (an impractical condition), the base of the weir was limited to a convenient length and for a very small height the opening was rectangular.

For water, the minimum height of the rectangular portion was suggested to be about $1/8'$. Pratt derived the equation of the curved portion of the Sutro-weir, but no experimental data was presented to support his findings. The main object was to determine the shape of the weir such that the discharge is proportional to the head above a given datum plane Fig. 1 b. He gave the relationship:

$$X = b \left(1 - \frac{2}{\pi} \tan^{-1} \sqrt{\frac{y}{a}} \right)$$

$$\text{and } Q = C a^{1/2} b \sqrt{2g} \left(h - \frac{a}{3} \right)$$

where x = width of weir at a point.

b = length of weir crest.

y = height of a point above the rectangular portion.

a = height of the rectangular portion.

Q = the rate of discharge.

h = the head over the weir crest.

E. Soucek, H. E. Howe, and F. T. Mavis⁽⁵⁾ in (1936), provided experimental verification for the accuracy of a similar type of weir. They conducted tests on eleven Sutro weirs. The tests showed that there was a close relationship between the coefficient of discharge and the geometrical proportions of the weir.

S. A. Greely and W. E. Stanley⁽⁶⁾ in (1942), stated that it was impossible to make base of Sutro-weir of infinite length and they

suggested that the weir be cut off at some width and the rejected area placed below the theoretical crest Fig. 1 c.

O. V. P. Stout (⁷), stated that for practical reasons, the curve of the Sutro-weir must be terminated at some point giving a crest of finite length b. The area of the weir opening rejected outside at A and A₁ Fig. 1 d is compensated for by two small rectangular orifices at A and A₁ equal to the rejected area, the centres of the orifices being placed at the level of the centroid of the cut out area.

CONDITION OF NEW TESTS

The new tests were made for a maximum flow of 13 lit./sec. and maximum head of 45 cm., the rectangular portion at the bottom was assumed to be 1.0 cm. depth. The weir was designed according to Rettger's (⁸) equation :

Equation of the curved outline is

$$x^2 y = c$$

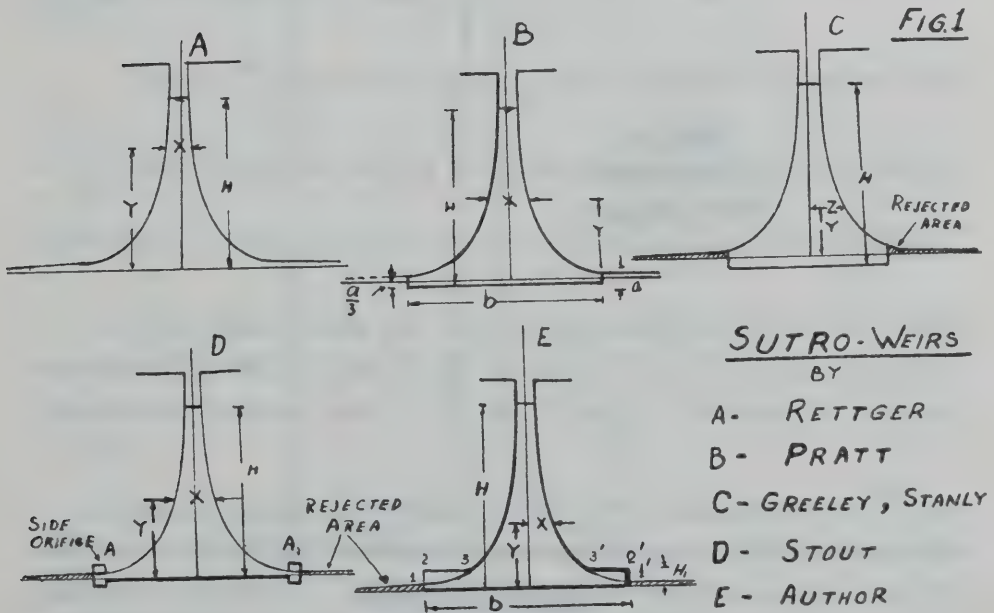


FIG 1

and equation of flow is

$$Q = K_1 H \left(\frac{\pi}{2} - \sqrt{\frac{H_1}{H}} \right)$$

where x = width of weir at any point in cm.

y = depth of the point above crest.

c = constant.

H = Head over weir.

H_1 = height of rectangular portion.

K_1 = constant.

$$= 2 Cd \sqrt{2gc}$$

MODIFIED
SUTRO - WEIR
INVESTIGATION

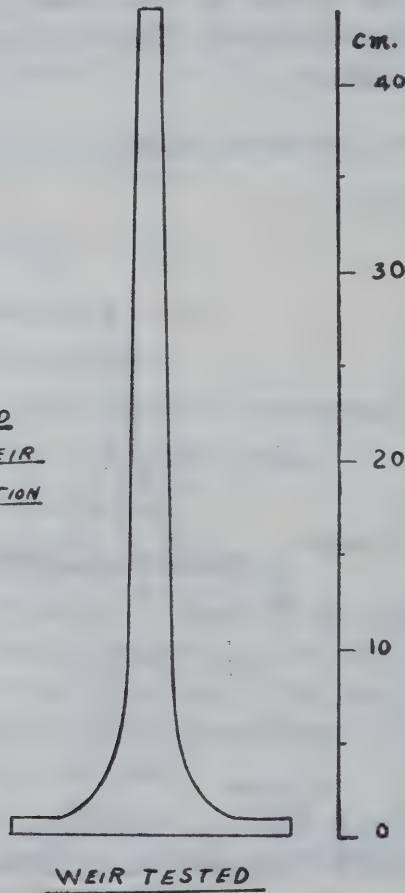


FIG. 1'

Cd is the coefficient of discharge and has to be assumed as 0.62 for the design according to Cornell tests (⁸).

According to the figures assumed above it was found :

$$K_1 = \frac{13,000}{45 \left(\frac{\pi}{2} - \sqrt{\frac{1}{45}} \right)} = 203.4$$

$$\text{and } C = \frac{K_1^2}{4 Cd^2 2g}$$

$$= \frac{(203.4)^2}{4 (0.62)^2 \times 2 \times 981} = 13.71$$

Hence the equation of the curved outline of the weir was

$$x^2 y = 13.71$$

The only point that now remains unobtained in the design is the rectangular portion of the weir and the following ways were adopted before, in taking that rectangular portion of the weir :

1. Rittger⁽³⁾ suggested that if $\frac{H_1}{H}$ is so small that the square root of $\frac{H_1}{H}$ in the equation of flow he obtained may be neglected in comparison with $\frac{\pi}{2}$, then the rejected area in this case could be neglected.

2. The rejected area has to be compensated for simply by placing two equal area 1, 2, 3, and 1', 2', 3', at the top of the curve at cut. The hatched rejected area (in figure) is equal to the area 1, 2, 3, and 1', 2', 3'.

Generally speaking, the area of the curve bounded by the X-axis and the parallel line 2—2' is the same as the area of the rectangular portion of the weir. Here, the theoretical and the actual crest are coincident and the full weir area can be utilised.

Or area of weir between $y = 0$ and $y = H_1$

$$\text{area} = \int_0^{H_1} 2x \, dy$$

$$\text{but } x = \sqrt{\frac{c}{y}} = \frac{m}{\sqrt{y}}$$

$$\text{area} = \int_0^{H_1} 2 \frac{m}{\sqrt{y}} \, dy$$

$$\text{for } H_1 = 1 \text{ cm.}$$

$$\begin{aligned} \text{area} &= 4\sqrt{c} = 4\sqrt{13.17} \\ &= 14.84 \text{ sq. cm.} \end{aligned}$$

or the length of crest $b = 14.84 \text{ cm.}$

DESCRIPTION OF APPARATUS

The apparatus was arranged as shown in Fig. 2, water was drawn from the constant level over-head tank through the 4' pipe system to the apparatus. The flow was regulated by means of a special globe valve, and the diffuser at the pipe and together with the horizontal strainer above it assist the steadiness of flow.

The flow through the weir was collected into the tank at the bottom of the apparatus, then left it through a swivelling chute to the measuring tank or to waste. A gauge tube was mounted in the front of the apparatus to show the water level or the pressure-head inside the apparatus. The point gauge fixed to apparatus side from inside is to establish datum for the weir crest. The weir shape was cut out in the $\frac{1}{4}$ " brass plate by means of a mechanical saw and the sides were made sharp-edged by hand filing.

EXPERIMENTAL OBSERVATIONS

The crest level was located on the head scale board simply by adjusting the inside gauge point to the crest level by the aid of a straight edge and a spirit level and filling the tank with water to that level and reading the scale gauge to give the crest level reading.

Tests were then made under the following conditions :

Range of head (over crest) : from 2.14 to 39.3 cm.

Range of rate of discharge : from 0.708 to 13.07 lit/sec.

Number of tests : 41

The theoretical discharge was based on Rettger's⁽³⁾ formula for the flow :

$$Q = k_1 \left(\frac{\pi}{2} - \sqrt{\frac{H_1}{H}} \right) H$$

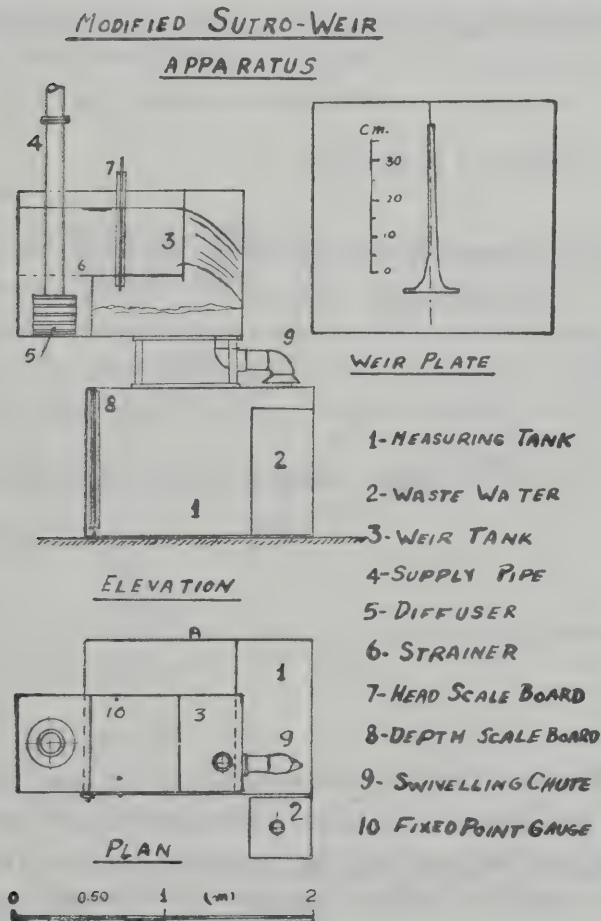


FIG. 2

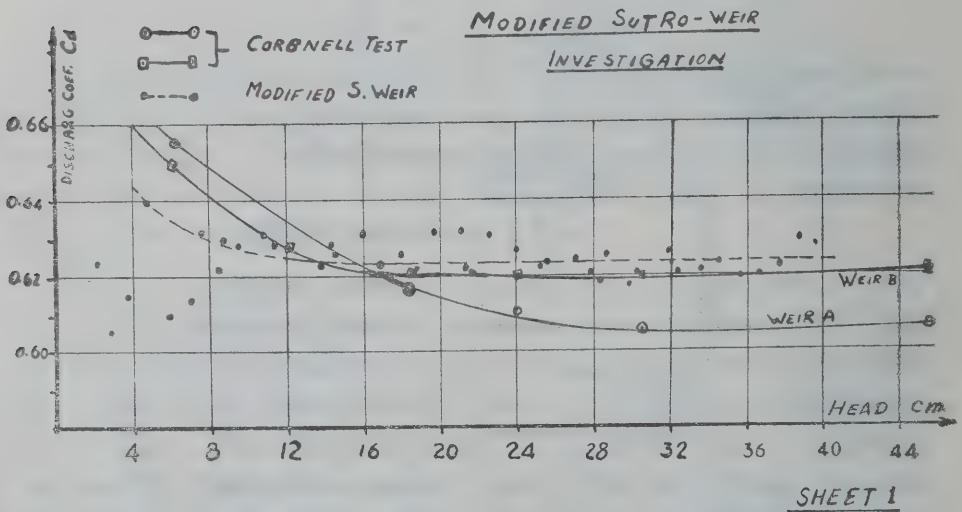
in which the value of $\frac{H_1}{H}$ is so small that its square root is negligible in comparison with $\frac{\pi}{2}$. The departure from proportionality is represented by the ratio :

$$\frac{\sqrt{\frac{H_1}{H}}}{\frac{\pi}{2}} = 0.637 \sqrt{\frac{H_1}{H}}$$

Thus, even when $\frac{H_1}{H} = 0.022$, the percentage departure from proportionality is about 9.5 % which is expected to be improved by compensation for the rejected area. The formula for the flow is :

$$Qt = K_1 \frac{\pi}{2} H = 0.53 H$$

where Qt in lit/sea ; H in cms.



The coefficient of discharge was calculated for each head and the average value was found to be 0.624. The coefficient of discharge Cd was plotted against the head and the curve is shown on (sheet No. 1) On the same sheet the values obtained by Coronell test⁽⁸⁾ were plotted for weir A and weir B. In comparing the results, the new values of Cd were consistent with those obtained by Coronell.

COMPARISON BETWEEN THE ACTUAL AND IDEAL CASES

The theoretical equation of flow is

$$Q_t = 0.53 H \quad (H \text{ in Cm.}; Q_t \text{ in lit/Sec.})$$

The ideal equation of flow is

$$Q_1 = 0.624 \times 0.53 H = 0.331 H \quad (\text{lit./sec.})$$

Let Q = actual measured discharge in lit/sec.

The ideal equation of flow was plotted, taking the head as ordinate and the discharge as abscissae, to be the ideal flow curve. The observation points were plotted on the same sheet and it was noticed that they nearly follow the ideal curve. Sheets 2 and 3 show the ideal curve and the observation points.

Curve Fitting :

To get the equation of the straight line to lie evenly amongst the observation points the method of zero sum⁽⁹⁾ was adopted. We arrange the observations in order of H and divide them into two equal sets. We then add up the Q 's and the H 's of each group now, the following equations have to be solved :

$$\Sigma_1 Q = \Sigma_1 H. a + \frac{1}{2} n b \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\Sigma_2 Q = \Sigma_2 H. a + \frac{1}{2} n b \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where $\Sigma_1 Q = 74.14$

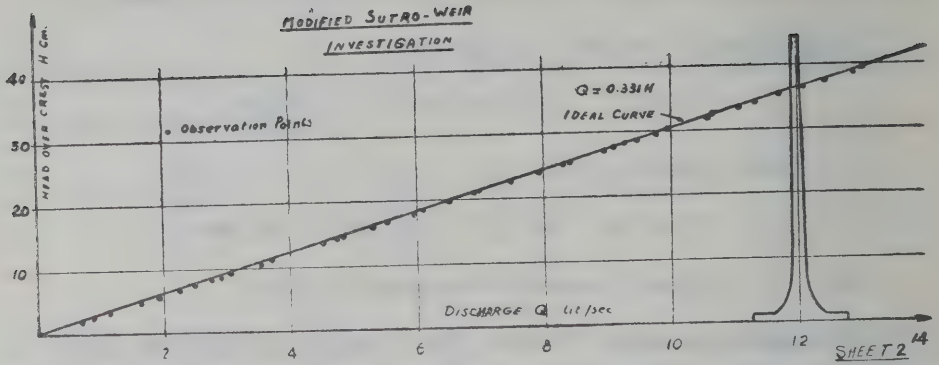
$$\Sigma_2 Q = 205.19$$

$$\Sigma_1 H = 223.77$$

$$\Sigma_2 H = 621.27$$

n = number of observation points a and b are constants in equation :

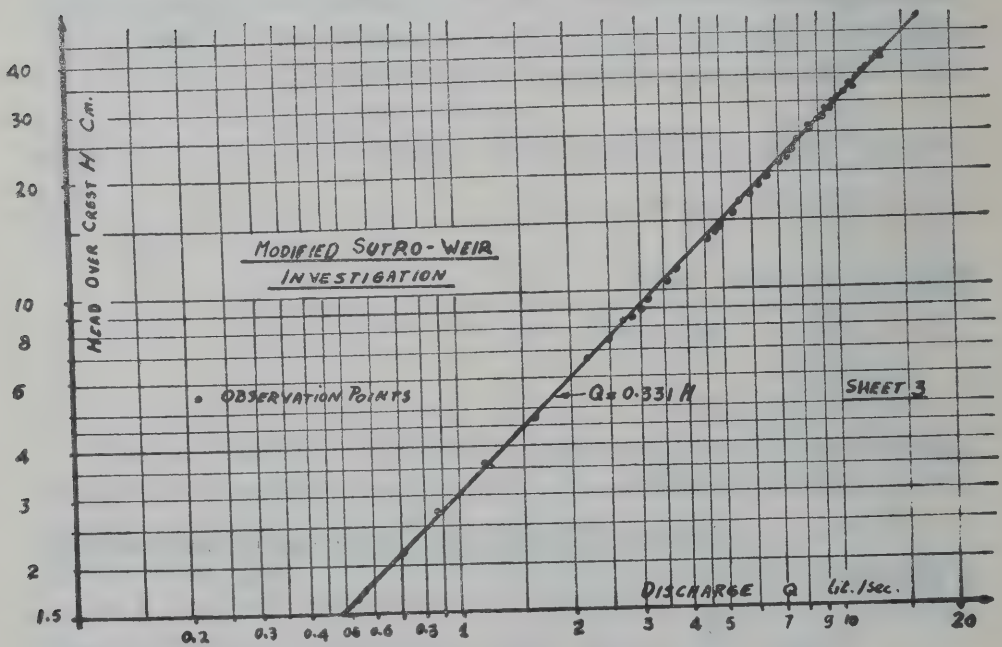
$$Q_{II} = a H + b$$



$$\text{or } 74.14 = 223.77 a + \frac{41}{2} b \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$205.19 = 621.27 a + 20.5 b \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\therefore a = 0.33 \text{ and } b = 0.0146$$



The equation of the straight line to be evenly amongst the observation points is :

$$Q_{II} = 0.33 H + 0.0146 \quad . \quad . \quad . \quad . \quad . \quad . \quad (a)$$

where Q_{II} is in lit/sec. and H is in cms.

The discharge was calculated for each head from the ideal equation :

$$Q_I = 0.331 H$$

the maximum percentage difference between the calculated and observed discharge was found to be 3.11 % at low heads while the minimum percentage difference was zero and it should be noticed that the average percentage difference was 0.73 %

Again, the discharge was calculated for each head from the equation of the straight line whose equation is (a) : the maximum percentage difference between the calculated and observed discharge was found to be 4.39 % at low heads while the minimum was 0.09 % and the average percentage difference was 0.82 % .

The ideal straight line equation and the equation of the straight obtained by the zero-sum method gave nearly the same discharge under the same head, the maximum percentage difference between the ideal Q_I and the straight line Q_{II} was 1.84 at lowest head and the minimum percentage difference was zero while the average percentage difference was 0.24 %

Effect of Small Error in Measurement of Head :

The head H over the weir crest may have an error $\pm \Delta h$ where Δh is the limit of absolute error for H that can be obtained. It is now required to find out the corresponding limit of absolute error ΔQ for the discharge Q .

If H changes from $(H - \Delta h)$ to $(H + \Delta h)$ and accordingly Q changes from $(Q - \Delta Q)$ to $(Q + \Delta Q)$ we can say that

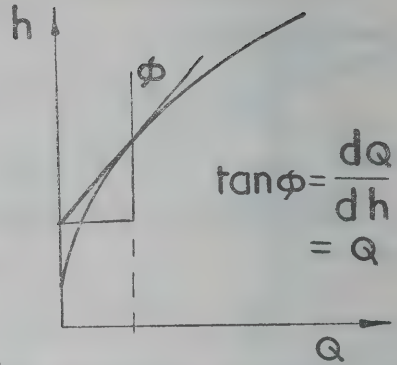
$$\Delta Q = \frac{dQ}{dh} \cdot \Delta h$$

From Curve

$$\tan \phi = \frac{Q}{x}$$

or
$$x = \frac{Q}{\tan \phi} = \frac{Q}{Q'}$$

$$= \frac{Q}{\frac{\Delta Q}{\Delta h}} = \frac{Q}{\Delta Q} \cdot \Delta h$$



Let
$$R_R = \text{relative error in } Q$$

$$= \frac{\Delta h}{Q} \tan \phi = \frac{\Delta h}{x}$$

or
$$R_Q = \frac{Q'}{Q} \Delta h$$

applying this theory to the weir under discussion we can say that :

The ideal equation is

$$Q_I = 0.331 H \quad Q_I \text{ in lis./sec, } H \text{ in cm.}$$

$$Q_I = \frac{d}{dQ} (Q_I) = 0.331$$

In measurement of head over the crest, the readings were taken to the nearest millimetre. In other words, the limit of absolute error in H was 1 mm. where H is measured in cm. i.e. $\Delta h = 1 \text{ mm.} = 0.10 \text{ cm.}$

$$R_Q = \frac{0.331}{Q_I} \cdot \Delta h$$

$$= \frac{0.331}{Q_I} \times 0.1 \times 100 \%$$

This will be the equation for the relative percentage error in measurement of discharge for a constant limit of absolute error in H of $\Delta h = 0.10 \text{ cm.}$ The relative percentage error R_Q was calculated for each discharge and the calculated values were plotted against Q_I as shown in (Sheet No. 4). The maximum percentage error was 3.31 % at

lowest discharge while the maximum percentage error was about 0.25%. This result, which gave nearly the same value for the maximum percentage error and the maximum percentage difference between the actually measured discharge and the calculated ideal discharge (3.11%), is a good verification for the theory of error in measurement. !

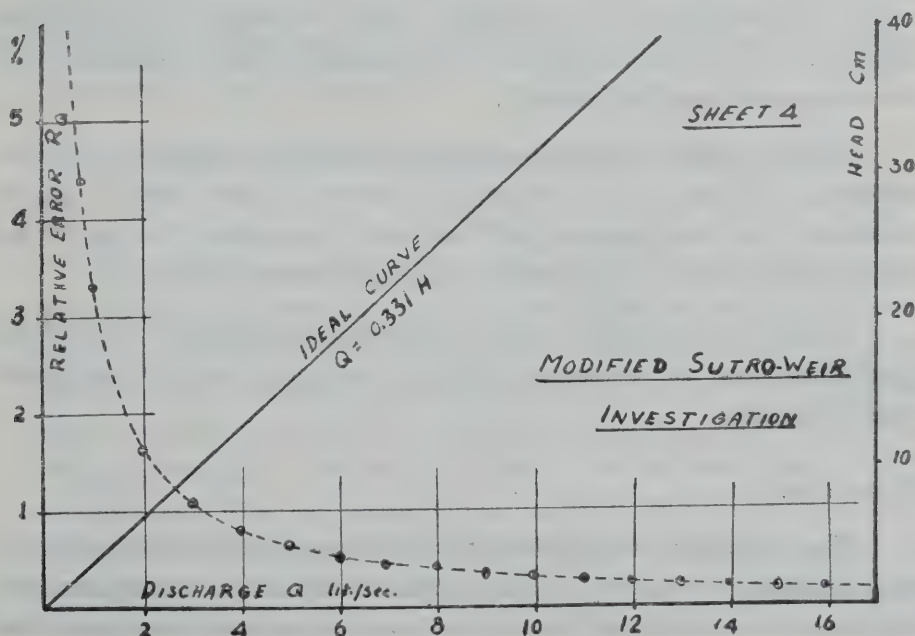
The Study of the Attainment of Fixed Water Level:

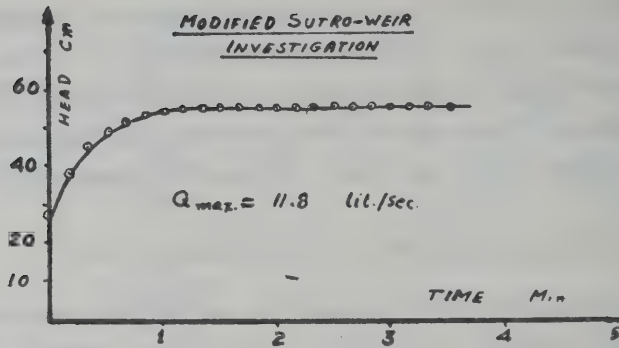
The proportional flow weir tank was set to work and the discharge was changed as quickly as possible to 11.8 lit/sec. causing the head to rise from 27 cm. and the head was recorded every 10 sec. a curve for the head in cm. and time in minutes was plotted as shown in (Sheet No. 5).

Discussion of Results:

From the results previously exhibited the following points should be cleared up in this discussion:

1. For such weir, the ideal curve or straight line is practically coincident with the mean straight line obtained by the zero-sum method if applied to the observation points.





SHEET 5

2. Due to the fact that the observation points practically lie on the ideal flow straight line also on the mean straight (lying evenly amongst them,) it could be said that they form a linear relationship between the head and the discharge within the full range of the usable head.

3. The theory of error in measurement gave a good forecast for the maximum percentage error expected for the ideal equation of flow and the results obtained were practically the same.

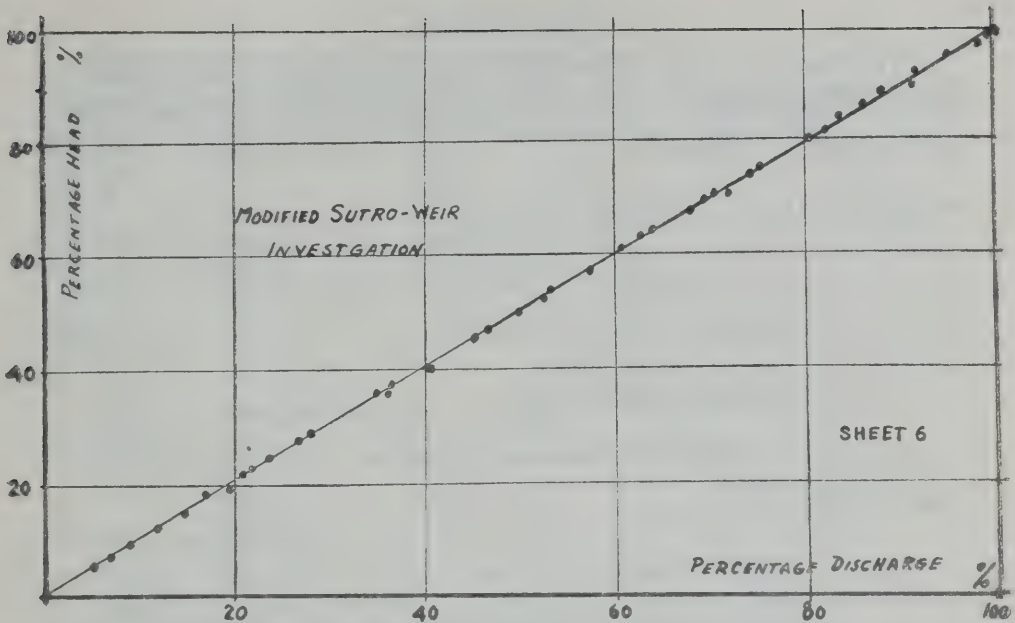
4. The coefficient of discharge of such weir was nearly constant except at low heads, and generally speaking it was consistent with the values obtained by Coronell.

5. If a curve for the percentage head (as ordinate) against percentage discharge (as abscissa) is plotted for the weir, it can be deducted that the proportional flow weir is suitable for nearly the full range of head. This curve is shown in (Sheet No. 6).

6. For the proposed weir, the head was measured above the theoretical crest or in other words the theoretical and actual crests were coincident.

Conclusion :

The proposed modification for the Sutro-weir which is simply compensating for the rejected area by introducing a rectangular area equivalent to area of the Sutro-weir bounded by the theoretical crest



and a parallel datum at a height H_1 which should be as small as practically possible, this modification proved to give good result for a very nearly linear relationship between the head and discharge within the full range of head over the weir crest. The error in measurement of the head and discharge can be minimised by improving the measuring instruments.

II.—DETERMINATION OF DISCHARGE COEFFICIENTS

SYNOPSIS

The author proposed ⁽¹²⁾ a modification for Sutro-Weir which is compensating the rejected area by introducing an area equivalent to the area of the weir bounded by the theoretical crest and a parallel datum at a small height H_1 , this modification proved to lead to good results for a very nearly linear relationship between the head and the discharge within the full-range of head over the weir crest. In this investigations, fourteen weirs of the modified type were tested in the Hydraulics Laboratories of the Faculty of Engineering of the University of Khartoum-Sudan, with a view of obtaining their discharge coefficients.

CONDITIONS OF TESTS

The tests were made for maximum flow of about 21 lit./sec. and maximum head of 30 cms. The weirs were designed according to Rettger's⁽³⁾ equations: equation for the outline of the weir,

$$x^2 y = c,$$

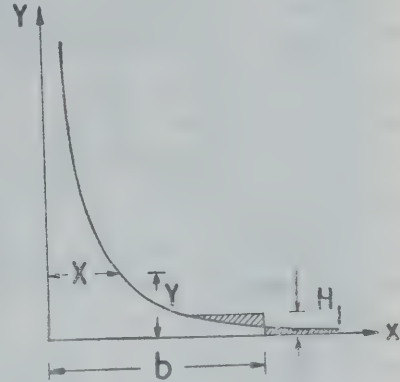
and the flow is give by the equation,

$$Q = K_1 H (\pi/2 - \sqrt{H_1/H})$$

where H_1 = height of the rectangular portion

Area of curve between $y = 0$ and $y = H_1$ is given by,

$$A = \int_0^{H_1} x dy = \int_0^{H_1} \sqrt{c/y} dy$$



As H_1 is small with respect to H , the theoretical flow equation is given by

$$Q_t = \pi/2 K H$$

where $K = \sqrt{2gc}$

Table 1 gives all data for the tested weirs

TABLE 1

Weir No.	Value of c	H_1 cms.	b cms.	weir shape	Q t/H
1	57.880	0.61	19.52	Non—Symmetric	0.529
2	130.524	0.61	29.28	Non—Symmetric	0.795
3	229.820	0.61	38.80	Non—Symmetric	1.055
4	63.838	0.61	40.87	Symmetric	1.112
5	57.880	1.50	12.42	Non—Symmetric	0.529
6	130.524	1.50	18.45	Non—Symmetric	0.795
7	229.820	1.50	24.39	Non—Symmetric	1.055
8	63.838	1.50	26.09	Symmetric	1.112
9	57.880	2.50	9.60	Non—Symmetric	0.529
10	130.524	2.50	14.40	Non—Symmetric	0.795
11	229.820	2.50	19.15	Non—Symmetric	1.055
12	63.838	2.50	20.20	Symmetric	1.112
13	229.820	4.00	15.125	Non—Symmetric	1.055
14	229.820	6.00	12.35	Non—Symmetric	1.055

x and y are in cms.

Q t in lit./sec. and H in cms.

DESCRIPTION OF APPARATUS

Figure 3 shows the arrangement of apparatus used. Water from a constant supply is allowed to run through the rectangular flume 11 and then over the weir plate 1 held at the end of the flume. The sluice valve 4 is to regulate the flow. The perforated outlet pipe 5, the stilling basin 10 and the wire-mesh strainers 3 are to ensure steadiness of flow. The tapping 6 is especially designed to give the average depth in the flume upstream the weir, and the point guage 8 fixed to the side of the flume is to fix the crest level and give the zero reading on the head scale board 9 for the measurement of the water surface elevation above the crest. The flow over the weir was collected, through a swivelling chute, into the under-ground measuring tank whose capacity is about 14,000 litres. The water temperature was measured near the weir plate. Weir plates were cut out of sheet iron $3/32$ inch thick. The boundaries of each opening were carefully marked, the opening was then cut out roughly with a jig saw and accurately filed to the required shape. The plates were securely braced on the downstream side to prevent any possibility of distortion in shape. The weir plate was bolted to the face of the flume with countersunk bolts and all rough edges in the approaches were carefully smoothed with modeling clay. In each test the weir crest was located approximately 39 cms. above the floor. The outlines of the weirs tested is to quarter scale on figures 4, 5, 6, and 7.

EXPERIMENTAL OBSERVATIONS

The crest level was located on the head scale board simply by adjusting the inside guage point to the crest level by the aid of a straight edge and a spirit level and then filling the flume with water to that level and reading off the scale board to give the zero or the crest level reading. For each test the water surface was given enough time to get steady before taking the reading and the water collected during the one test was over 2,000 litres or for a period of 5 minutes with small rates of discharge. Photo 1 and 2 illustrate how the nappe is coming out of a non-symmetrical and a symmetrical weir.

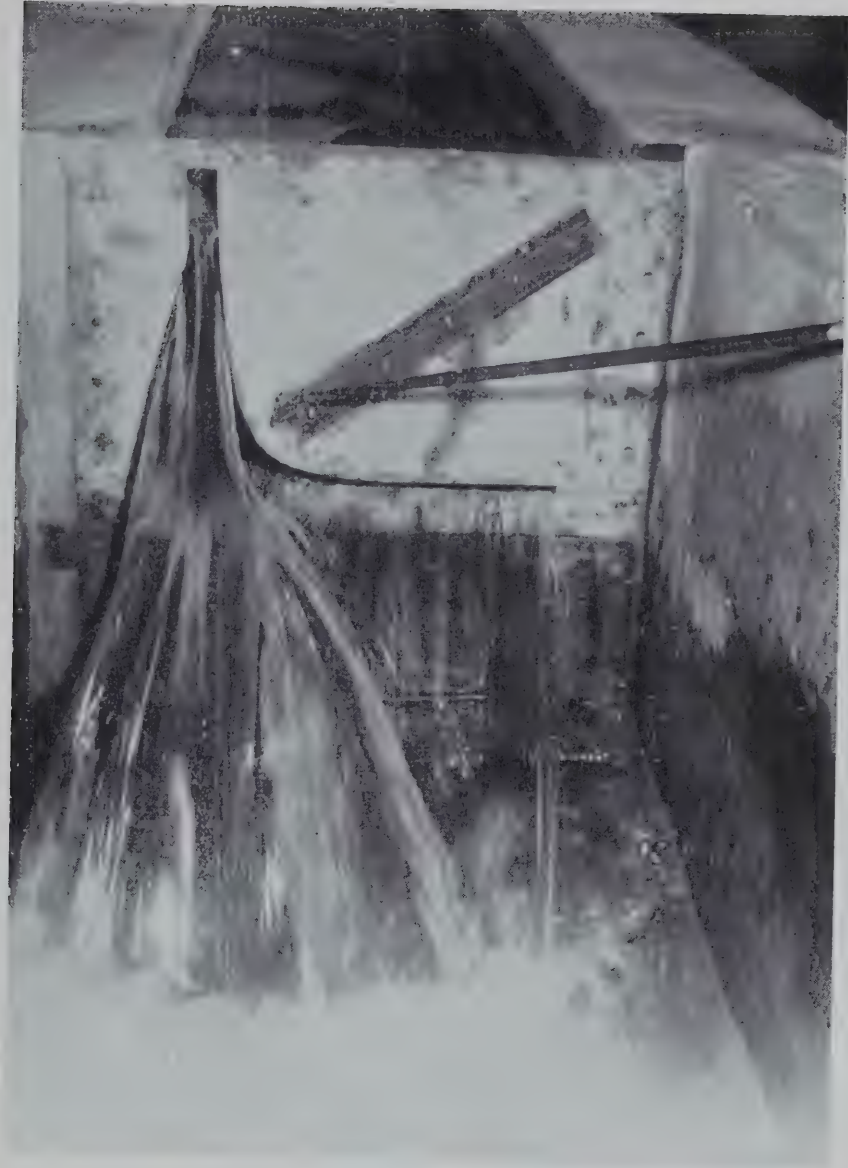


PHOTO 1
Non—Symmetrical Weir Under Test.



PHOTO 2
Symmetoical Weir Under Test.

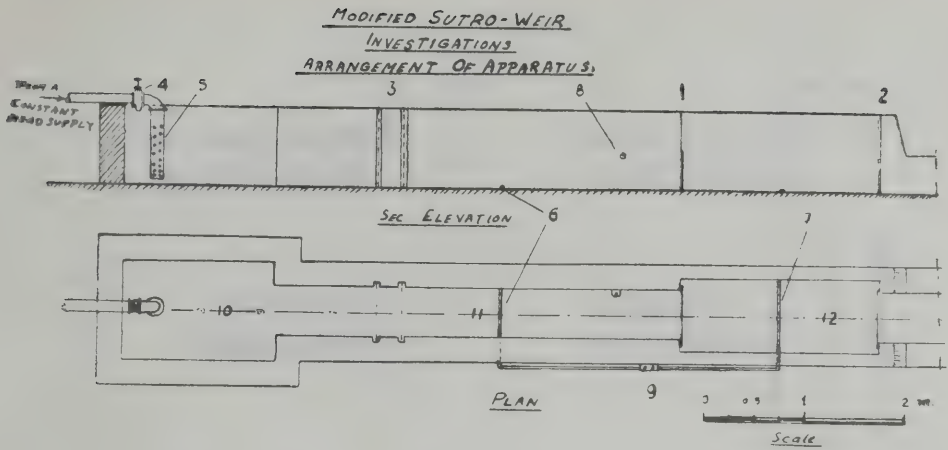


FIG. 3

- | | |
|---------------------------|----------------------|
| 1. Weir Plate | 7. Downstream Taping |
| 2. Downstream Gate | 8. Point Gauge |
| 3. Wire Mesh Baffles | 9. Head Scale Board |
| 4. Sluice Valve | 10. Stilling Chamber |
| 5. Perforated Outlet Pipe | 11. Testing Flume |
| 6. Upstream Taping | 12. Downstream Flume |

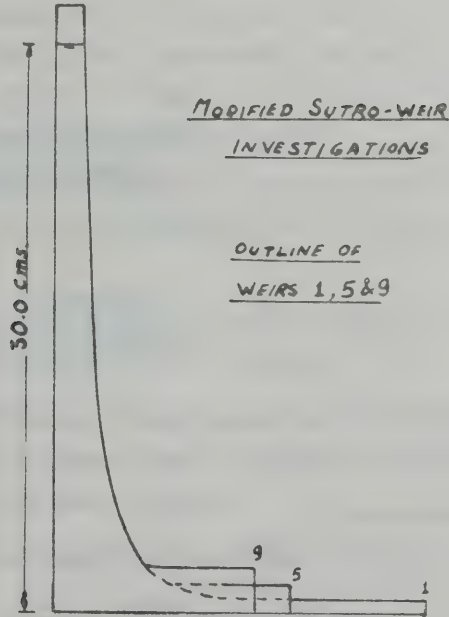


FIG. 4

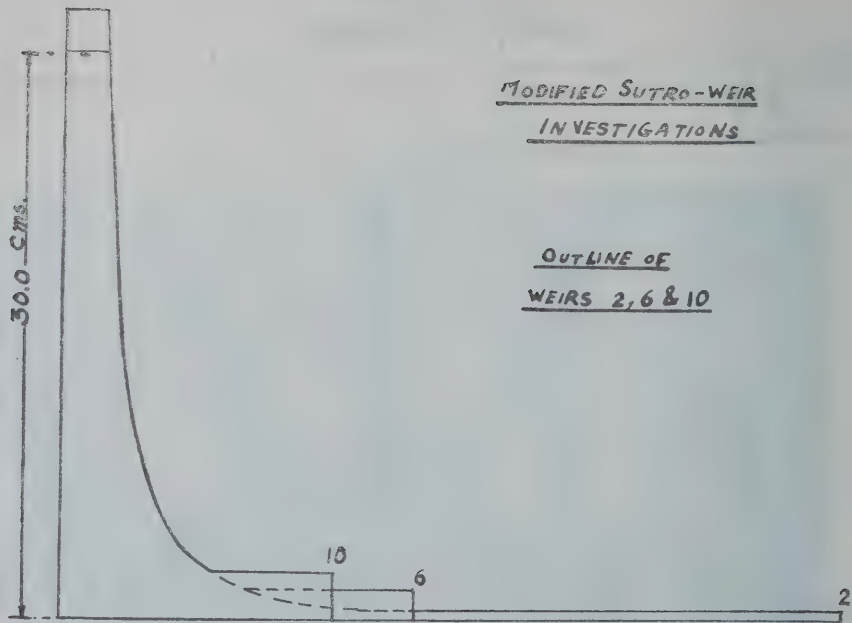


FIG. 5

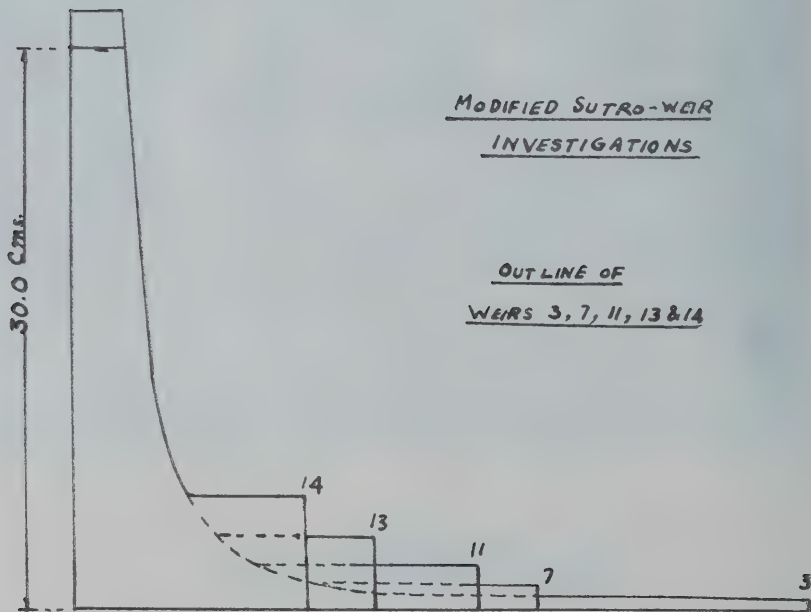


FIG. 6

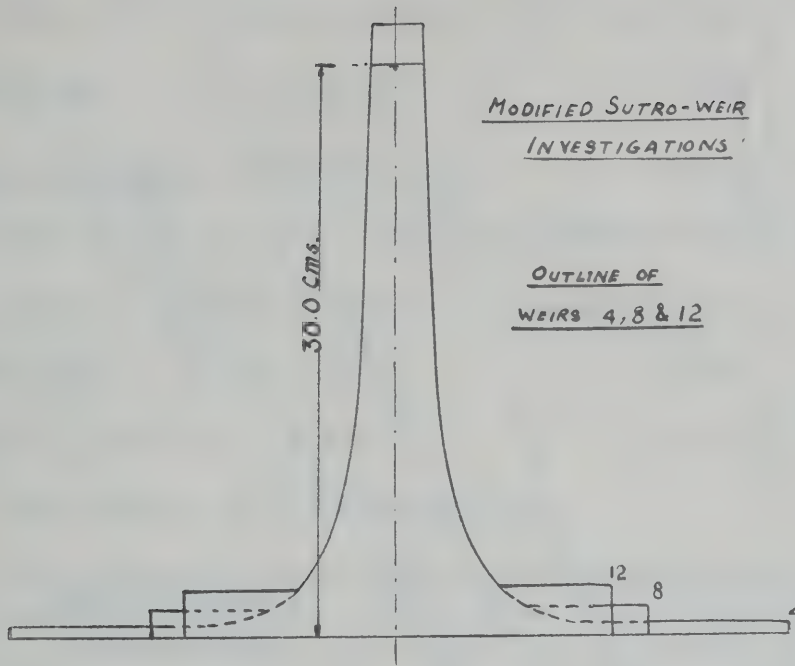


FIG. 7

The observations for the head above the weir crest in cms. were plotted against the actually measured discharges for the fourteen different weirs as given in sheet 7. From the curves already obtained in sheet 7 the following can be summarised :

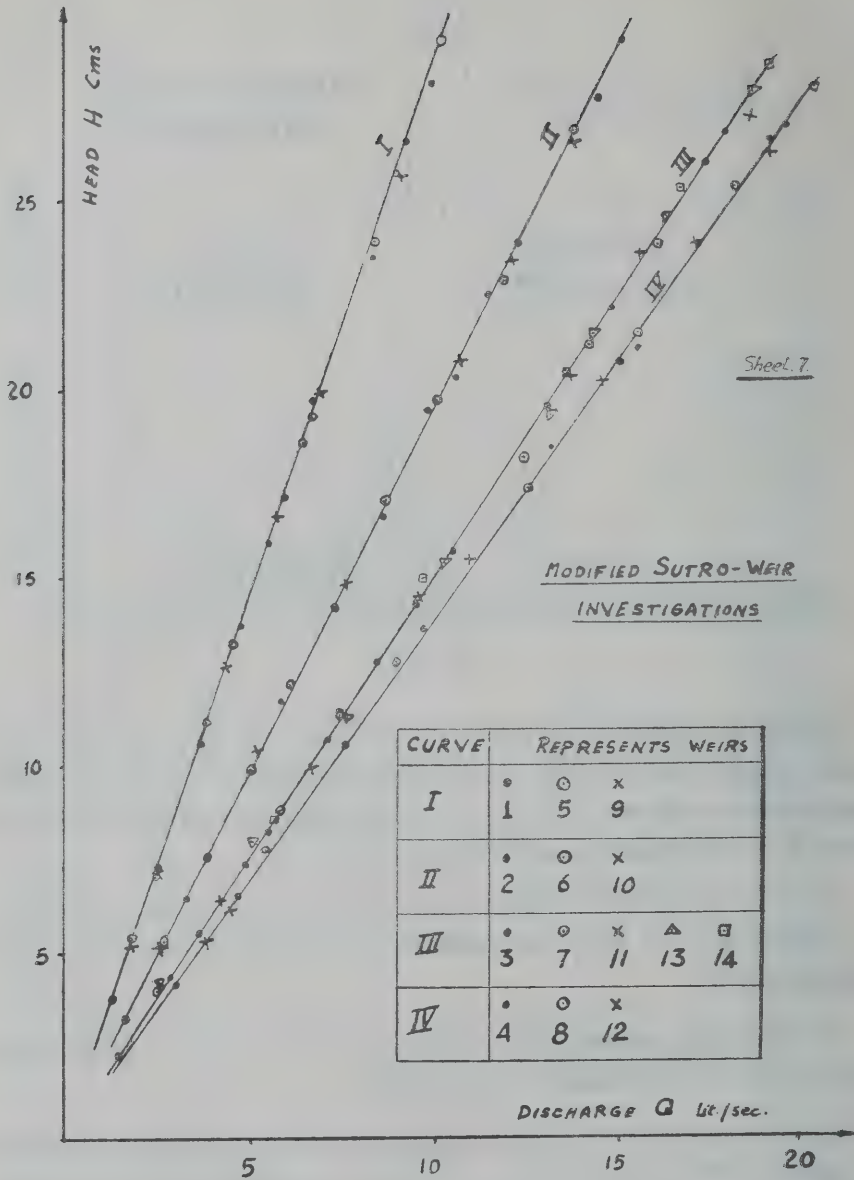
(a) The observation points of weirs having the same curved outline equation and the same theoretical flow equation have practically to lie on one curve.

(b) The observation points again ensure the linear relationship for the head and discharge of such weirs.

It can be said that the weirs tested are to form between themselves four groups. Each group has a common equation for the curved outline and the same flow equation.

GROUP I Includes the weirs Nos. 1, 5, and 9, they have :

flow equation $Q_t = 0.529 H$ lit./sec.
outline equation $x^2 y = 57.880$ x and y in cms.



GROUP II Includes the weirs Nos. 2, 6, and 10, they have :

flow equation : $Q_t = 0.895 H$ lit./sec.

outline equation $x^2 y = 130.524$ x and y in cms.

GROUP III Includes the weirs Nos. 3, 7, 11, 13, and 14, they have :

flow equation $Q_t = 1.055 H$ lit./sec.

outline equation $x^2 y = 229.820$ x and y in cms.

The first three groups are of the non-symmetrical weir type.

GROUP IV. Includes the weirs No 4, 83, 12, they have :

flow equation $Q_t = 1.112 H$ lit./sec.

outline equation $x^2 y = 63.838$ x and y in cms.

These weirs are of the symmetrical type

The evaluated discharge coefficients were plotted against the head for each group and they are given on sheets (8, 9, 10, and 11) for groups I, II, III, IV respectively. Again, the total average value for the discharge coefficients for each group were plotted against the value of C in the equation of the curved outline as given on sheet 12.

DISCUSSION OF RESULTS

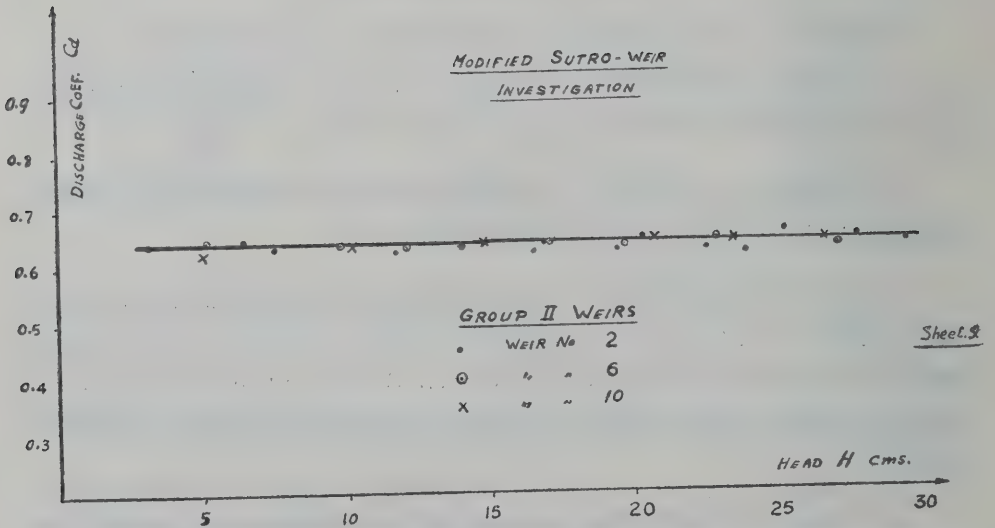
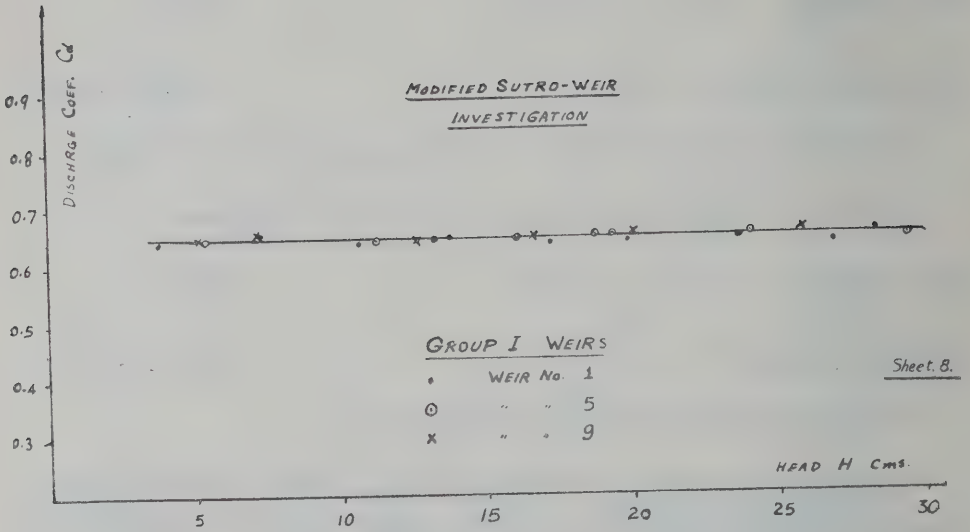
Briefly the above tests lead to the following conclusions :

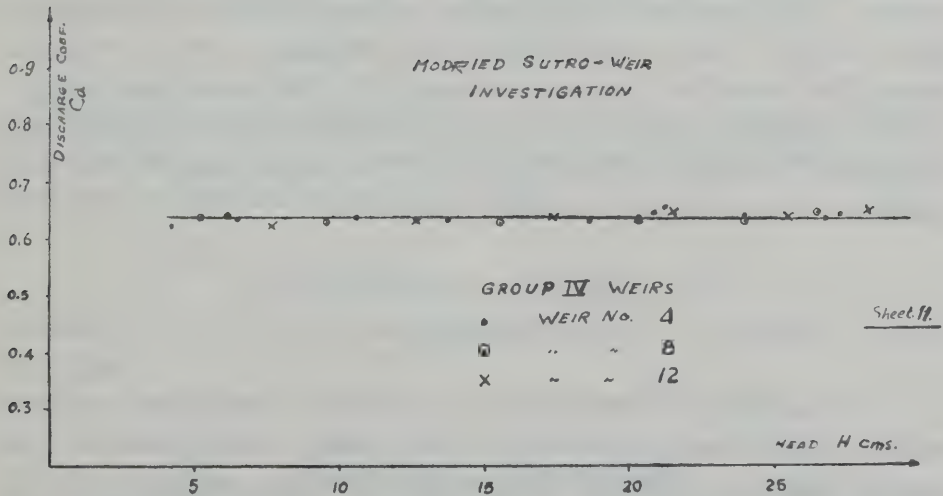
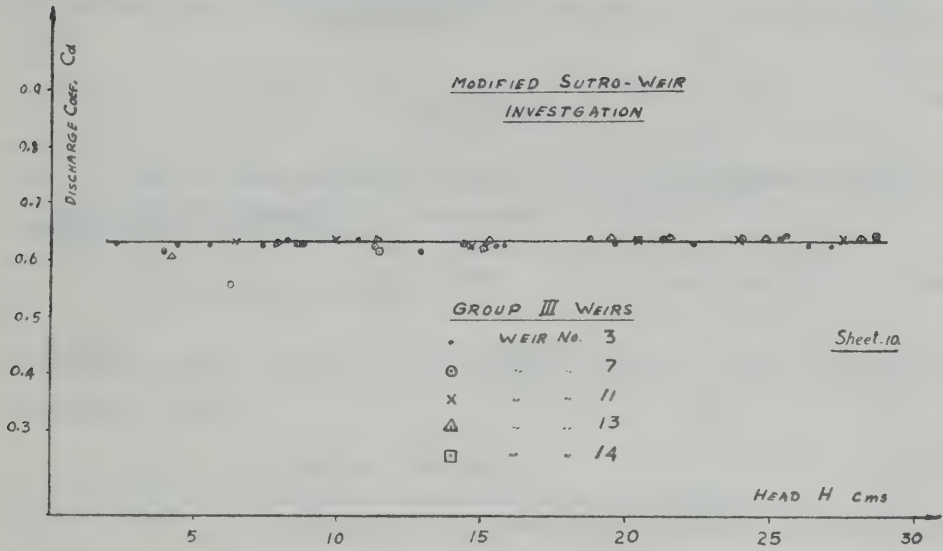
1. Figure 7 ensures that for the whole range of head up to twice H_1 the linear head discharge relationship exists.

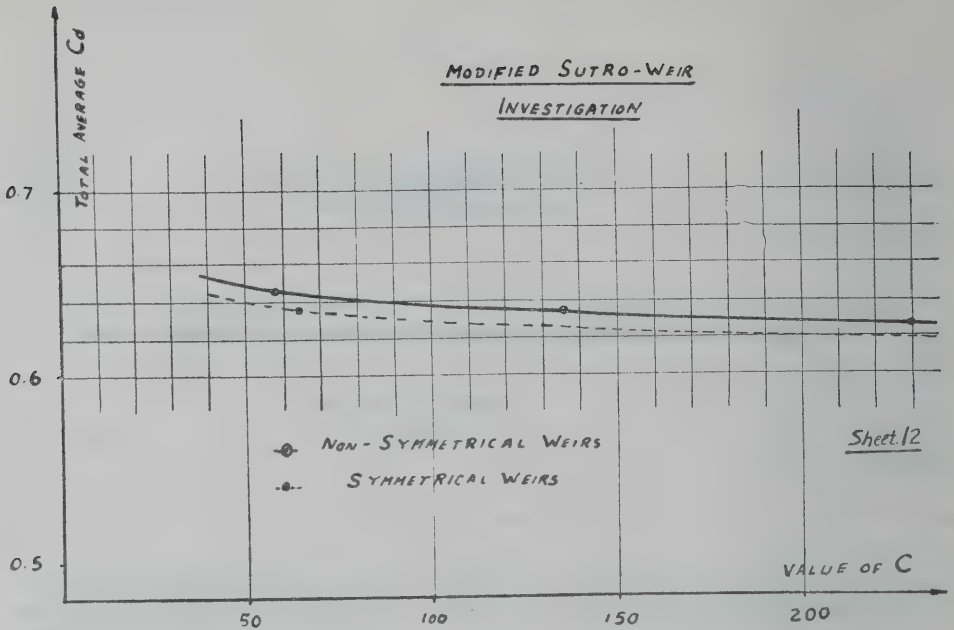
2. Curves 8 to 11 show how the discharge coefficient is nearly constant for the full range of the tested heads.

3. Curve in fig. 12 show that the discharge coefficient varies with the value of C in the curved outline equation or in other words varies with the geometrical shape of the weir. It can be said, from fig. 12, that the smaller the value of C the bigger the discharge coefficient will be.

5. Values of Cd for the symmetrical weirs were one per cent less than those for non-symmetrical weirs of the same value of C.







CONCLUSIONS

This investigation leads to the fact that the discharge coefficient for the Modified Suro-Weirs depends mostly upon their geometrical shape. The dimensions of the bottom rectangular portion has a slight effect on C_d . The curve given on Fig. 12 can be used in obtaining a reasonable value for C_d for a given value of C which is accurate enough for the design purpose. The discharge coefficient for the symmetrical weirs may be taken as one per cent less than that for a non symmetrical weir having the same value of C .

III.—EFFECT OF SUBMERGENCE

SYNOPSIS

The question of submergence is important in weir flow analysis whatever the shape of the weir is. For sharp crested weirs of the normal shape the effect of submergence has been studied by J. R. Villemonte⁽¹⁰⁾ and also by F. T. Mavis⁽¹¹⁾. For the Suro-Weirs of the modified form as suggested by the author⁽¹²⁾, the flow under

submerged conditions is the object of this paper. Tests on four different weirs of this type were made in the Hydraulics Laboratories of the Faculty of Engineering of the University of Khartoum-Sudan.

INTRODUCTION

In case of sharp-crested weirs of any type become submerged, Villemonte (¹⁰) formula :

$$Q = Q_1 (1 - s^n)^{0.385}$$

provides a simple method of evaluating the flow under submerged condition, where :

Q = the actual discharge,

Q_1 = the discharge at the same head without submergence

S = the submergence ratio, $= h d / h u$,

$h d$ = downstream elevation above crest,

$h u$ = upstream elevation above crest.

n = the power to which the head is raised in the free flow

equation : $Q = K H^n$

If the formula is applicable for the modified Sutro weir tested, the exponent n in the Villemonts formula may be put as equal to one and the formula can be put in the form :

$$Q/Q_1 = (1 - S)^m$$

where m for all other types of weirs tested by Villemonte is equal to 0.385.

DESCRIPTION OF APPARATUS

The same apparatus described in part II was used for this study. Figure (3) shows the apparatus under submerged condition.

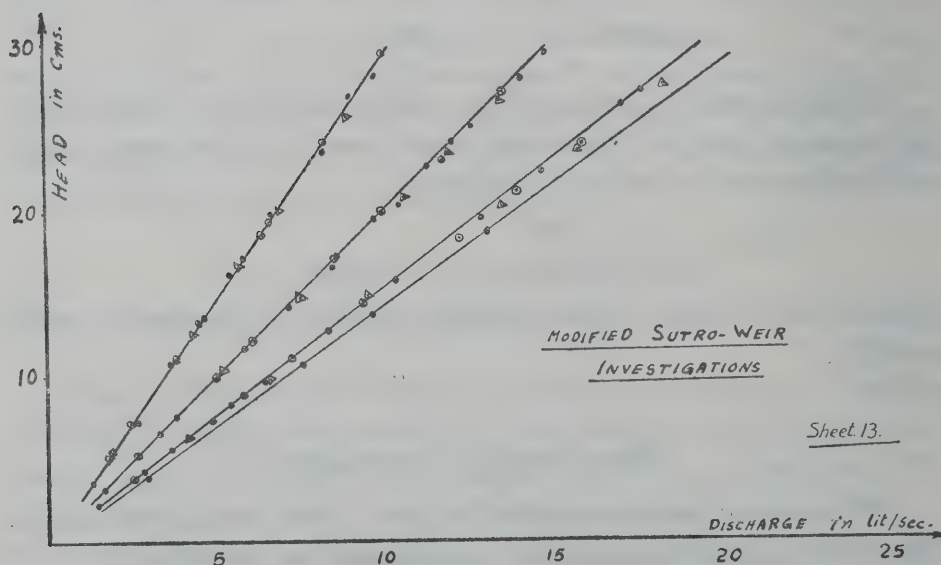
EXPERIMENTAL OBSERVATIONS

The weirs tested were made according to the following data :

TABLE 1

Weir No	1	2	3	4
Shape of weir	Non-Sym.	Non-Sym.	Non-Sym.	Symm.
Curved outline $X^2Y =$ } (X and Y in cms.) }	57.880	130.524	229.820	63.838
Rect. portion { H_1 (cm.)	0.61	0.61	0.61	0.61
{ b (cm.)	19.52	29.28	38.80	40.87
Flow Equation : { K	0.529	0.795	1.055	1.112
$Q = Cd K H$ } lit/sec cm. { av. Cd	0.644	0.635	9.627	0.635

The weirs were calibrated and the calibration charts are given in sheet (13) while the average discharge coefficients are given in table (1) above.



With a certain free flow, the downstream water surface level was raised little by little until it submerged the weir crest. When the steady conditions were attained, the upstream and downstream water surface elevations above the crest were read off the head scale board. Again, the downstream water surface level was raised for a second reading and so on. Tests were made under the following conditions :

TABLE 2

Weir No.	No. of discharges tested	% age of submergence		Range of flow lit/sec		No. of tests made
		from	to	from	to	
1	7	4.8	84.0	5.433	9.033	31
2	6	10.9	85.2	3.180	12.810	31
3	7	6.9	93.1	5.458	16.778	35
4	7	2.3	92.1	4.367	17.056	30

Photo (3) shows the non-symmetrical weir No. 3 with the nappe disturbance at a low % age of submergence.

Photo (4) shows the symmetrical weir No. 4 with the nappe disturbance at a low % age of submergence.

Photos (5) and (6) show the submerged flow of the same weirs Nos. 3 and 4 under a big % age of submergence.

Sheets (14), (15), (16), and (17) are sets of curves for the upstream and downstream water surface elevations above the crest for the four weirs Nos. 1, 2, 3, and 4 respectively. The values for the free flow Q_1 were taken from the head discharge curves resulting from the calibration of the weirs and given on sheet (13).

The quantities $\log. \left(\frac{Q}{Q_1} \right)$ were plotted against the quantities $\log (1 - S)$ to obtain the value of the exponent m . Nearly all the observation points practically lie on one straight line whose inclination was found to be 0.386 which agrees to a great extent with the value obtained by Villemonte⁽¹⁰⁾.

RESULTS AND CONCLUSIONS

The results already exhibited show that the sharp crested modified Sutor-weir when becomes submerged, its flow can be evaluated to a great degree of accuracy by the formula:

$$Q = Q_1 (1 - S)^{0.386}$$

also it is possible to find out the submergence ratio S for a given discharge ratio $\frac{Q}{Q_1}$, vice versa.

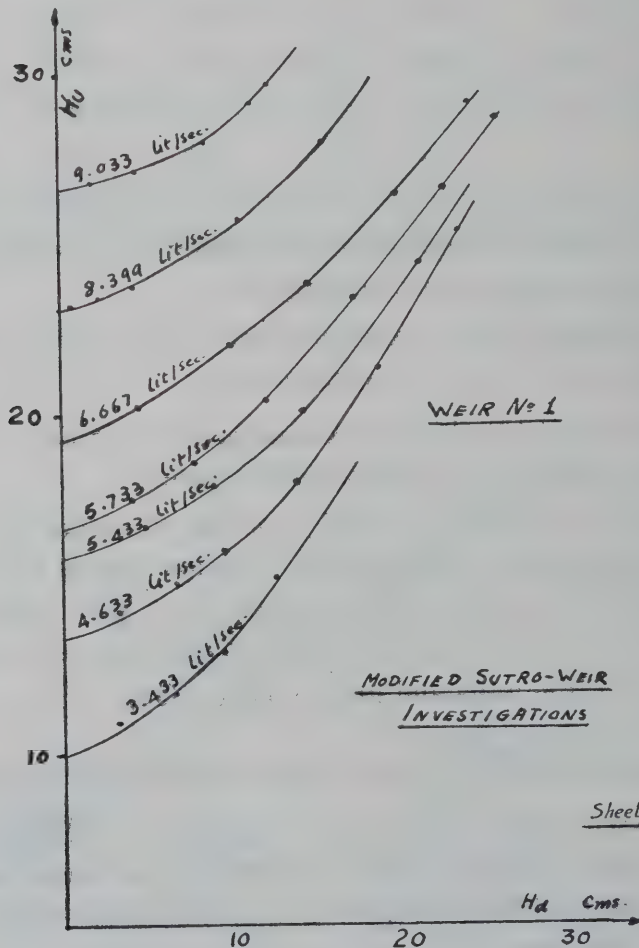




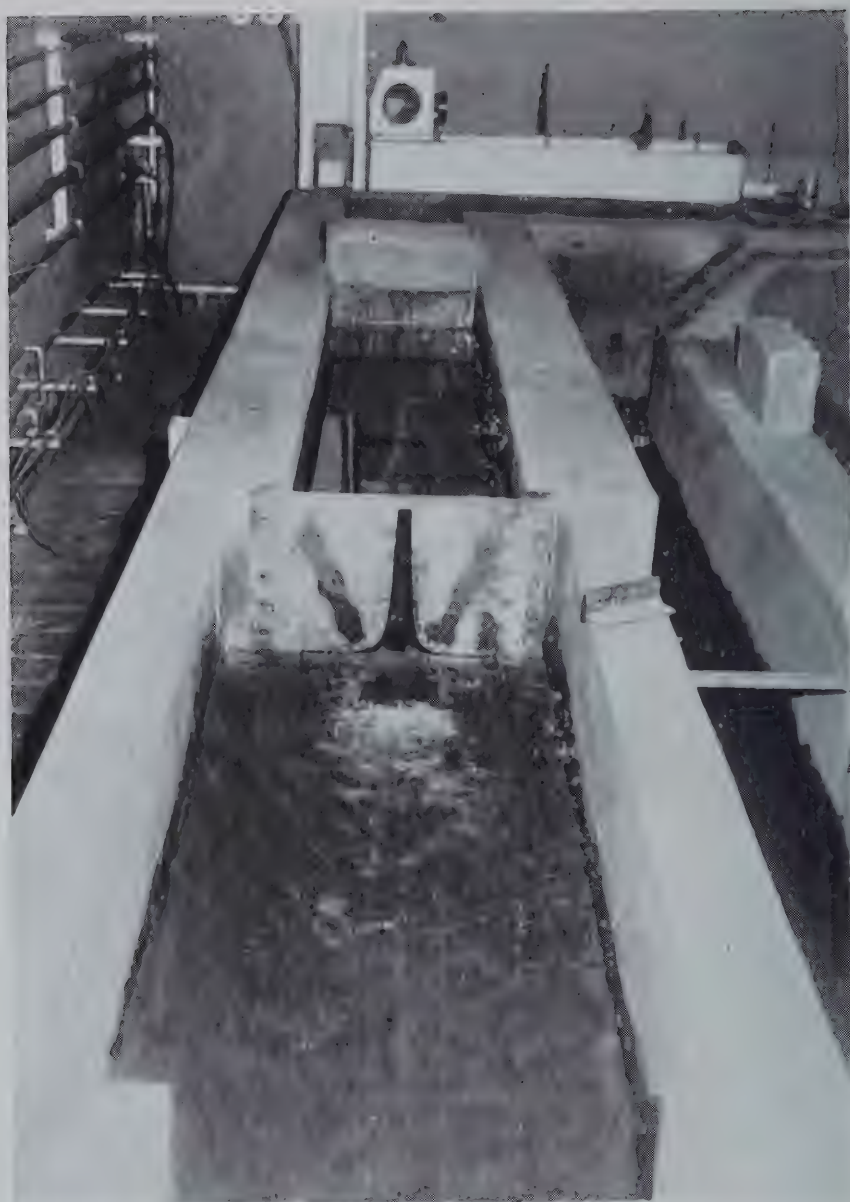
PHOTO 3



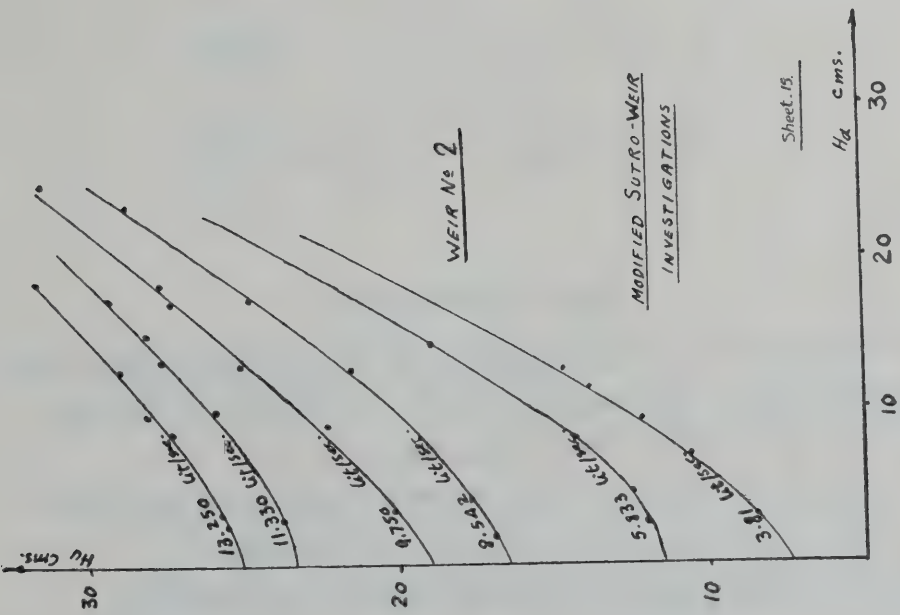
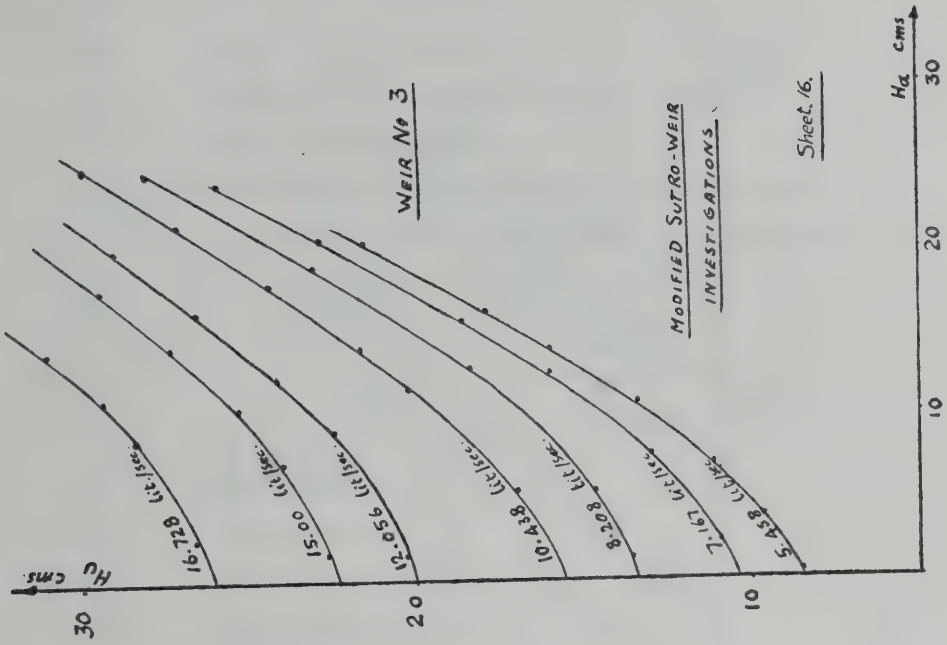
РНОТО 4

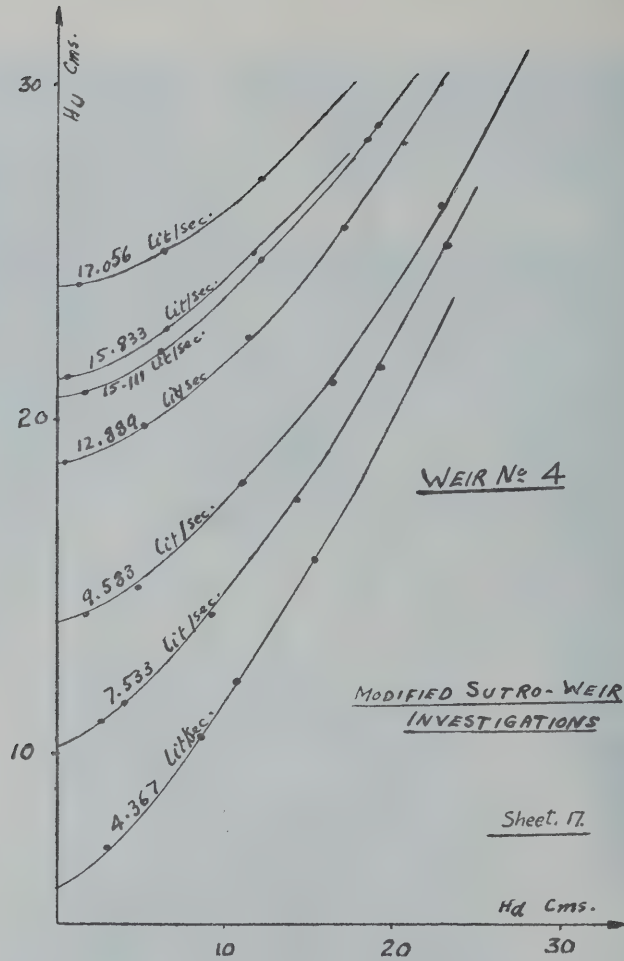


РНОТО 5



РНОТО 6





KEY OF SYMBOLS USED

The letter symbols adopted for use in this paper define where they first appear, and are listed here for convenience of reference.

x, X = width of weir at a point.

y, Y = height of a point above a certain horizontal datum.

b = length of crest of a weir or a constant.

a = height of rectangular portion or a constant.

c = constant.

- Q == actual rate of discharge.
 Q_1 == a certain discharge.
 H_1 == height of rectangular portion or constant.
 h, H == head over the crest.
 h_u, H_u == upstream water surface elevation above crest.
 h_d, H_d == downstream water surface elevation above crest.
 s, S == submergence ratio H_d/H_u .
 k, K == constant.
 m, n == number or power.
 c_d == coefficient of discharge.
 Q_t == theoretical discharge.
 Q_i == ideal discharge.
 Q_{11} == rate of discharge.
 Q' == first derivative of Q .
 ΣQ == sum of Q 's.
 ΣH == sum of H 's.
 Δh == limit of absolute error in H .
 ΔQ == limit of absolute error in H .
 R_Q == limit of relative error in Q .
 g == acceleration due to gravity.
 ϕ == angle.

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CONCENTRATED FORCE PROBLEMS IN THE TRANSVERSE BENDING OF ELASTICALLY RESTRATINED PLATES

BY

W. A. BASSALI⁽¹⁾ and R. H. DAWOUD⁽²⁾

This paper deals with the effect of an elastic boundary constraint upon the deflexion, bending and twisting moments and shearing forces in thin isotropic plates acted upon by concentrated normal forces. The complex variable method for solving the biharmonic equation is here applied to two problems. In the first we deal with the circular plate subjected to a general boundary condition including the rigidly clamped and hinged (simply supported) edges. The second problem is that of an infinite plate elastically restrained at an inner circular boundary, its outer edge being free. Typical curves of deflexion, moments and shear forces are plotted.

NOMENCLATURE

The following nomenclature is used in the paper

C	Circular boundary of plate.
c	Radius of circular boundary.
w	Deflexion of mid plane of plate.
D	Flexural rigidity of plate.
x, y	Cartesian coordinates.

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r, θ Polar coordinates.

η Poisson's ratio.

$-\mu = M_n/M_s$ Ratio between bending moments at any point on C.

λ, γ, m, k Restraining parameters.

$K; \left. \begin{array}{l} \delta_1, \delta_2, \delta_3; \\ \gamma_1, \gamma_2, \gamma_3; \end{array} \right\}$ Dimensionless quantities defining deflection of mid surface of plate, bending and twisting moments, shearing forces and vertical reaction; see equations [34] and [37].

M_n, M_s, M_r, M_θ Bending moments per unit length.

$M_{ns}, M_{r\theta}$ Twisting moments per unit length.

$Q_r, Q_\theta; V_r$ Shearing forces and vertical reaction per unit length.

$$\beta = (1 - \eta)/(1 + \eta) \quad , \quad \lambda = \beta(\mu - 1)/(\mu + 1) \quad ,$$

$$v = (1 + \lambda)/(1 - \lambda) \quad ,$$

$$m = 1/(1 - \lambda) = \frac{1}{2}(1 + v) \quad , \quad k = v - \eta = 2m - 1 - \eta$$

$$f, f'; R, \Theta; R', \Theta'; \omega, \omega'; \quad \text{see Fig. 2.}$$

$$z = x + iy = re^{i\theta} \quad , \quad \bar{z} = x - iy \quad , \quad Z = z - f \quad ,$$

$$Z' = f' - z$$

$$q = f/c = c/f' \quad , \quad \rho = r/c \quad , \quad \varepsilon = x/c$$

Other symbols are defined as they appear in the text.

INTRODUCTION

Methods using complex variables have been applied to plate problems by many writers. A particular form of the boundary condition defining certain types of constraint at the boundary of a thin isotropic plate has been introduced by the authors (1,2)⁽¹⁾. This condition assumes that the ratio between the boundary bending moments M_n and M_s is constant at all points of the boundary. In the special case of a circular boundary it is found that this condition agrees

(1) Numbers in parenthesis refer to the bibliography at the end of the paper.

with Reismann's condition (3) in which he assumed that the boundary of the plate is restrained against rotation according to a linear law, *i.e.* the slope of a line normal to the boundary and tangent to the meridian plane at the boundary is taken proportional to the radial bending moment.

The problem of a transverse isolated load at any point of a clamped circular plate was solved in different methods by Clebsch (4), Michell (5) and Flügge (6). Schmidt (7) obtained a solution for a clamped circular plate uniformly loaded over an eccentric circle. A series solution for the hinged (simply supported) circular plate under a concentrated force was given by Föppl (8). The method of complex potentials was applied by Dawoud (9) to solve the problem of an eccentric isolated load under certain boundary conditions. Applying Muskhelishvili's Method, Washizu (10) got the same results for the clamped and simply supported boundaries. Reismann (3) used an infinite series solution in polar coordinates to consider the effect of an elastic boundary constraint upon the deflexion, moments and shearing forces at any point of a thin circular plate acted upon by a concentrated normal force

In this paper the complex variable method is first used to solve Reismann's problem and the same method is then applied to the problem of an infinite plate elastically restrained at an inner circular boundary, with outer edge free, and loaded by a transverse force at any point. A solution for the latter problem was obtained by Symonds (11) when the inner circular boundary is clamped while Dean (12) treated the case of an inner clamped elliptic boundary.

The general assumptions relating to the bending theory of thin plates are adopted. The two problems discussed show that the method of complex potentials represents a considerable gain in algebraic simplicity over previous methods.

Numerical results for a certain position of the load are presented in the form of typical curves of deflection, moments and shear forces.

CONDITIONS AND STRESSES ALONG A CIRCULAR BOUNDARY

The transverse displacement w at any point z of the mid plane of a thin elastic isotropic plate satisfies the biharmonic equation

$$\nabla^4 w = \frac{16}{\partial z^2 \partial \bar{z}^2} \frac{p}{D} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where p is the transverse load intensity and D is the flexural rigidity of the plate.

The general solution of Equation [1] is expressed in the form

$$w = 2 \operatorname{Re} [\bar{z} \phi(z) + \psi(z)] + W(z, \bar{z}) \quad . \quad . \quad . \quad (2)$$

where $\phi(z)$ and $\psi(z)$ are two analytic functions and $W(z, \bar{z})$ is a particular integral of [1].

Our first boundary condition is that the plate is supported such that $w = 0$ along its boundary C . The second boundary condition used here is that introduced by the authors in a previous paper (1, 2). Over a circular boundary $r = c$, this condition is

$$M_r/M_\theta = \text{constant}, -\mu, \text{ say} \quad . \quad . \quad . \quad . \quad (3)$$

In terms of ϕ , ψ and W this condition takes the form (1,2)

$$\operatorname{Re} \left[\lambda \left\{ z \phi'' + \frac{z^2}{c^2} (\psi'' + W'') \right\} - 2 \phi' - \frac{\partial^2 W}{\partial z \partial \bar{z}} \right] = 0 \text{ over } C, (4)$$

$$\text{where } \lambda = \beta (\mu - 1)/(\mu + 1), \quad \beta = (1 - \eta)/(1 + \eta), \quad (5)$$

and dashes denote derivatives with respect to z .

We also note here that in terms of the usual polar coordinates (r, θ) the boundary condition (3) reduces to

$$\frac{\partial^2 w}{\partial r^2} + \frac{\nu}{c} \frac{\partial w}{\partial r} = 0 \text{ over } r = c, \quad . \quad . \quad . \quad (6)$$

where $\gamma = (1 + \lambda)/(1 - \lambda)$ (7)

Over this circular boundary we have

$$\frac{M_r}{\eta - \gamma} = \frac{M_\theta}{1 - \eta \gamma} = - \frac{D}{c} \frac{\partial w}{\partial r} = \frac{D}{\gamma - 1} \nabla^2 w, \quad (8)$$

$$\frac{M_{r\theta}}{1 - \eta} = \frac{c Q_\theta}{\gamma - 1} = \frac{D}{c} \frac{\partial^2 w}{\partial \theta \partial r} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

If we now put $\gamma - \eta = k$, (10)
equation (8) shows that

$$M_r = \frac{k D}{c} \frac{\partial w}{\partial r} \quad \text{over the circular boundary} \quad . \quad (11)$$

This agrees with Reismann's condition which assumes that the slope of a line normal to the boundary and tangent to the meridian plane at the boundary is directly proportional to the radial bending moment.

In the problem of a thin circular plate under normal loading the dimensionless restraining parameter k varies from 0 for a simply supported boundary to ∞ for a clamped boundary so that from (10) we must have

$$\eta \leq v \leq \infty, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

and from (5) and (7) we get

$$-\beta \leq \lambda \leq 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

If an infinite thin plate is supported along an inner circular boundary subject to the condition (3), which reduces to (11), the parameter k varies from $-\infty$ to 0 (corresponding to the clamped and simply supported boundaries respectively), so that in this case we have

$$-\infty \leq v \leq \eta; \quad -\infty \leq \lambda \leq -\beta \quad \text{and} \quad 1 \leq \lambda \leq \infty. \quad (14)$$

ELASTICALLY RESTRAINED CIRCULAR PLATE UNDER AN ECCENTRIC ISOLATED NORMAL FORCE

Let a normal force F be applied at the point $Q = (f, 0)$, $f < c$, of a thin circular plate of radius c and boundary C and let $Q' = (f', 0)$ be the inverse of Q with respect to C so that $ff' = c^2$. If P is the point $z = re^{i\theta}$ of the plate, then with the notations of Fig. 1 we put

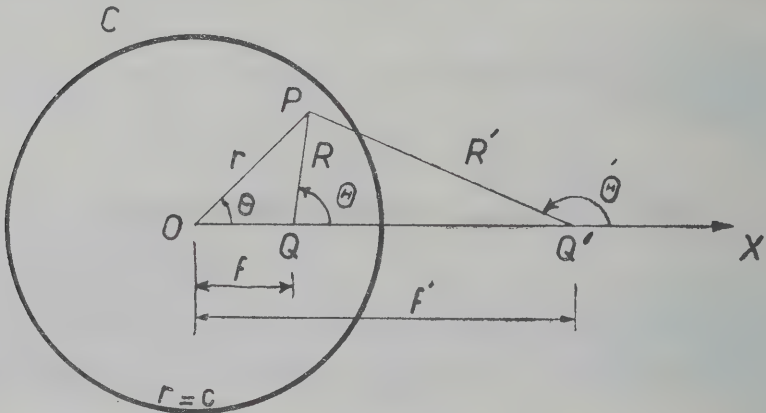


FIG. 1

$Z = z - f = R e^{i\Theta}$, $Z' = f' - z = R' e^{i(\Theta' - \pi)}$ (15)
so that for all points on C we have

$$|z| = c, \left| \frac{c Z}{f Z'} \right| = 1, \frac{f}{Z} = \frac{z}{Z'}, \frac{z}{Z} = \frac{f'}{Z'} \quad (16)$$

It has been shown (9) that if $w = 0$ over C , the analytic functions ϕ and ψ due to an isolated force F at $(f, 0)$ are of the form

$$\left. \begin{aligned} \phi(z) &= \alpha [z G(z) + Z \log(c Z / f Z')], \\ \psi(z) &= \alpha [c^2 G(z) + f z \log(c Z / f Z')], \end{aligned} \right\} \quad (17)$$

where $\alpha = F/16 \pi D$ and $G(z)$ is regular over the whole plate and is determined by the boundary condition.

Substituting from (17) in (4), taking $W(z, \bar{z}) = 0$, and using (16) we have

$$\operatorname{Re} [(1-\lambda)z G'(z) + G(z) + f'(1-\lambda)(1-q^2)/Z' + \frac{1}{2}\lambda(1-q^2)] = 0 \text{ on } C,$$

where $q = f/c = c/f'$.

This condition can be satisfied by equating the expression between the square brackets to iK' , where K' is a real constant that can be dropped since it contributes nothing to w . Putting

$$G(z) + \frac{1}{2}\lambda(1-q^2) = H(z), \quad z = tf', \quad 1-\lambda = 1/m, \quad (18)$$

we obtain the linear differential equation

$$dH/dt + mH/t + (1-q^2)/t(1-t) = 0. \quad (19)$$

Solving this equation we get

$$H(z) = (q^2 - 1) I_m(t) + K'' t^{-m}, \quad t = z/f',$$

where K'' is an arbitrary constant and

$$I_m(t) = t^{-m} \int_0^t \frac{t^{m-1}}{1-t} dt = \sum_{n=0}^{\infty} \frac{t^n}{n+m}, \quad |t| < 1. \quad (20)$$

When $\lambda \leq 1$, we have $m > 0$ and for $G(z)$ to be regular at $z = 0$, the constant K'' must be zero so that

$$G(z) = \frac{1}{2}(q^2 - 1) [\lambda + 2I_m(t)], \quad t = z/f'. \quad (21)$$

For a clamped boundary $\lambda = 1$, $m = \infty$ so that Equations (20), (21) and (18) give.

$$G(z) = \frac{1}{2}(q^2 - 1), \quad H(z) = 0.$$

These results for the clamped boundary may be obtained by solving (19) and (18) with $m = \infty$ and $\lambda = 1$.

The deflexion w is obtained on substitution from (21) and (17) in (2) and we find

$$w = \frac{F}{8\pi D} \left[R^2 \log \frac{cR}{fR'} + (c^2 - r^2)(1 - q^2) \right] \left\{ \frac{1}{2}\lambda + \sum_{n=0}^{\infty} \frac{(r/f')^n \cos n\theta}{n+m} \right\}. \quad (22)$$

This agrees with Reismann's result (3) on expanding $R^2 \log R'$ and noticing that $\lambda = (\eta + k - 1)/(\eta + k + 1)$, $2m = \eta + k + 1$ and that Reismann uses the symbol ν for Poisson's ratio (instead of our η). Numerical results as calculated by Reismann will be found in his paper (3).

Putting $\lambda = 1$ and $\lambda = \beta$ successively in (22) we get the solutions for the clamped and hinged (simply supported) boundaries respectively. Washizu (10) obtained the same results by applying Muskhelishvili's method.

LARGE PLATE ELASTICALLY RESTRAINED AT AN INNER CIRCULAR BOUNDARY

Consider an infinite thin plate with an outer free edge, bounded internally by a circular boundary C of radius c , and acted upon by a normal isolated force F at a point $Q = (f, 0)$, $f > c$. If $Q' = (f', 0)$ is the inverse of Q with respect to C so that $ff' = c^2$ (see Fig. 2), then the relations (15) and (16) will still hold here.

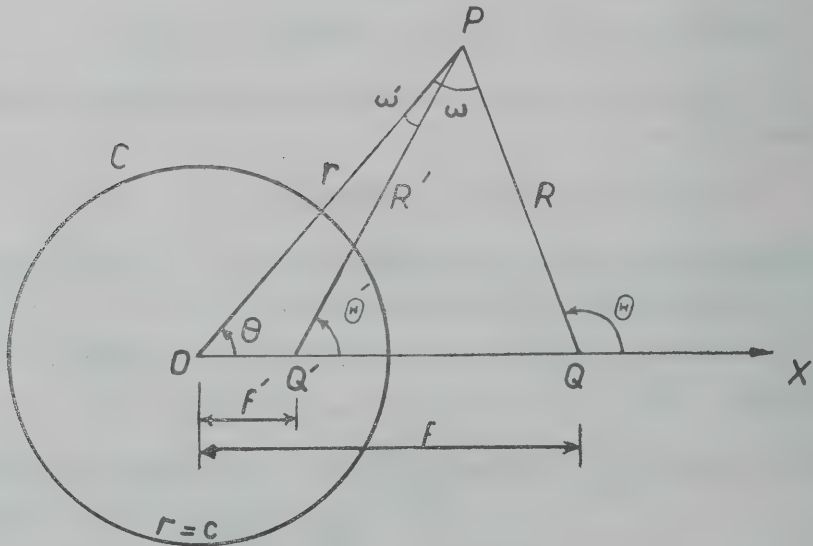


FIG. 2

For the analytic functions $\phi(z)$ and $\psi(z)$ let us assume tentatively that

$$\left. \begin{aligned} \phi(z) &= \alpha [z G(z) + Z \log(cZ/fZ')], \\ \psi(z) &= -\alpha [c^2 G(z) + fZ \log(cZ/fZ') + Ac^2 \log(z/c)], \end{aligned} \right\} \quad (23)$$

where $\alpha = F/16\pi D$, A is a real constant and $G(z)$ is regular over the whole plate.

Substituting in (2) taking the particular integral $W = 0$ we get

$$w = 2\alpha [(r^2 - c^2) \operatorname{Re} G(z) + R^2 \log(cR/fR') - Ac^2 \log(r/c)], \quad (24)$$

which satisfies the two requirements that $w = 0$ over C and that near the load point Q , the singular part of w is given by $\frac{F}{8\pi D} R^2 \log R$ as it should.

We have now to determine the function $G(z)$ to satisfy the boundary condition (4) which, on making use of (16), leads to

$$\begin{aligned} \operatorname{Re} [(1 - \lambda) z G'(z) + G(z) + \tfrac{1}{2} \lambda (1 - A - q^2) \\ + f' (1 - q^2) (1 - \lambda) / Z'] = 0 \text{ along } C, \end{aligned}$$

where $q = f/c = c/f'$.

This condition can be satisfied by equating the expression between the square brackets to iK' where K' is a real constant. It is however clear from [24] that a term iK' in $G(z)$ will lead to $w = 0$ everywhere so that K' can be taken zero. Putting

$$G(z) + \tfrac{1}{2} \lambda (1 - q^2 - A) = H(z), \quad z = f'/t, \quad 1 - \lambda = 1/m, \quad (25)$$

we obtain the differential equation

$$dH/dt - mH/t + (1 - q^2)/(1 - t) = 0. \quad (26)$$

$$\text{whence } H(z) = K' t^m + (q^2 - 1) J_m(t), \quad t = f'/z, \quad (27)$$

where K' is an arbitrary constant of integration and

$$J_m(t) = t^m \int_0^t \frac{t^{-m}}{1 - t} dt = \sum_{n=1}^{\infty} \frac{t^n}{n - m}, \quad |t| < 1. \quad (28)$$

From (25) and (7) we see that $m = \frac{1}{2}(1 + \gamma)$, and using (14) we have $-\infty \leq m \leq \frac{1}{2}(1 + \eta)$, so that in this case m does not take positive integral values (since $\eta < 1$).

It only remains to satisfy the condition that the outer edge of the infinite plate is free. This requires that $M_n \rightarrow 0$ and that also

$$V_n = Q_n - \frac{\partial M_{ns}}{\partial s} \rightarrow 0 \text{ as } |z| \rightarrow \infty. \text{ It is however obvious that}$$

for an infinitely large plate the solution is independent of the particular boundary curve of the outer edge. Taking this outer edge as a circle of very large radius we see that the previous conditions may be taken as

$$M_r \rightarrow 0 \text{ and } V_r = Q_r - \frac{1}{r} \frac{\partial M_{r\theta}}{\partial \theta} \rightarrow 0 \text{ as } |z| = r \rightarrow \infty. \quad (29)$$

The moments and shearing forces at any point of the plate are calculated from the formulae (2)

$$\left. \begin{aligned} M_r &= -2D(1 - \eta) \operatorname{Re} \left[z \phi''(z) + \frac{z^2}{r^2} \psi''(z) \pm \frac{2}{\beta} \phi'(z) \right], \\ M_{r\theta} &= -2D(1 - \eta) \operatorname{Im} \left[z \phi''(z) + \frac{z^2}{r^2} \psi''(z) \right], \\ Q_r - i Q_\theta &= -8D \frac{z}{r} \phi''(z). \end{aligned} \right\} \quad (30)$$

Substituting from (23), (25) and (27) in (30) we find that to satisfy the conditions (29) and to insure a single valued solution for fractional values of m we have

$$K'' = 0, \quad \lambda A = \lambda(1 - q^2) + 2 \log q. \quad (31)$$

Equations (25) and (27) will then give

$$G(z) = \log q + (q^2 - 1) J_m(f'/z) \quad (32)$$

Substituting from (31) and (32) in (23) and (24) we can express the functions ϕ and ψ and the deflexion w in the forms

$$\left. \begin{aligned} \phi(z) &= \alpha [z \log q + Z \log (cZ/fZ') \\ &\quad + (q^2 - 1) z J_m (f'/z)] , \\ \psi(z) &= -\alpha c^2 [\log q + (qZ/c) \log (cZ/fZ') \\ &\quad + (q^2 - 1) J_m (f'/z) + \{1 - q^2 \\ &\quad + (2/\lambda) \log q\} \log (z/c)] , \end{aligned} \right\} \quad (33)$$

$$\left. \begin{aligned} w &= K F c^2/D , K = K_1 + K_2 , \\ \text{where } 8\pi K_1 &= \frac{1}{2} (q^2 + \rho^2 - 2 q \rho \cos \theta) \times \\ &\quad \log \frac{q^2 + \rho^2 - 2 q \rho \cos \theta}{1 + q^2 \rho^2 - 2 q \rho \cos \theta} + (q^2 - 1 - \log q) \log \rho \\ &\quad + (\rho^2 - 1 - \log \rho) \log q , \\ 8\pi K_2 &= \frac{2}{1 - m} \log q \log \rho \\ &\quad + (q^2 - 1) (\rho^2 - 1) C_m (\rho , \theta) , \end{aligned} \right\} \quad (34)$$

$$q = f/c , \rho = r/c , C^m (\rho, \theta) = \sum_{n=1}^m \frac{(\rho q)^{-n} \cos n \theta}{n - m} . \quad (35)$$

It is now clear that the deflexion w given by (34) is a symmetrical function of the pair q, ρ so that the known reciprocal relation which can be written $w (q, \rho) = w (\rho, q)$ is satisfied and that also $w (\rho, \theta) = w (\rho, -\theta)$ as is expected from the symmetry of the problem.

It also appears that the part K_2 depends on the parameter $m = \frac{1}{1 - \lambda}$ representing the type of constraint at the boundary and that it vanishes for a clamped boundary for which $\lambda = 1, m = -\infty$. Thus for a clamped inner circular boundary our result for w agrees with that obtained by Symond (11) and by Dean (12).

From (34) we see that the deflexion coefficients K_1 , K_2 at the load point Q ($\rho = q$, $\theta = 0$) are given by

$$\left. \begin{aligned} 4 \pi K_1 &= (q^2 - 1 - \log q) \log q , \\ 4 \pi K_2 &= \frac{(\log q)^2}{1 - m} + \frac{1}{2} (q^2 - 1)^2 \sum_{n=1}^{\infty} \frac{q^{-2n}}{n - m} . \end{aligned} \right\} \quad (36)$$

Fig. 3 shows the deflexion profile along the axis of symmetry in the case $q = 2$, corresponding to different values of the restraining parameter m , Poisson's ratio being taken as $\frac{1}{3}$. In Fig. 4 the deflexion factor K at the load point ($\rho = q = 2$) is plotted against the parameter m .

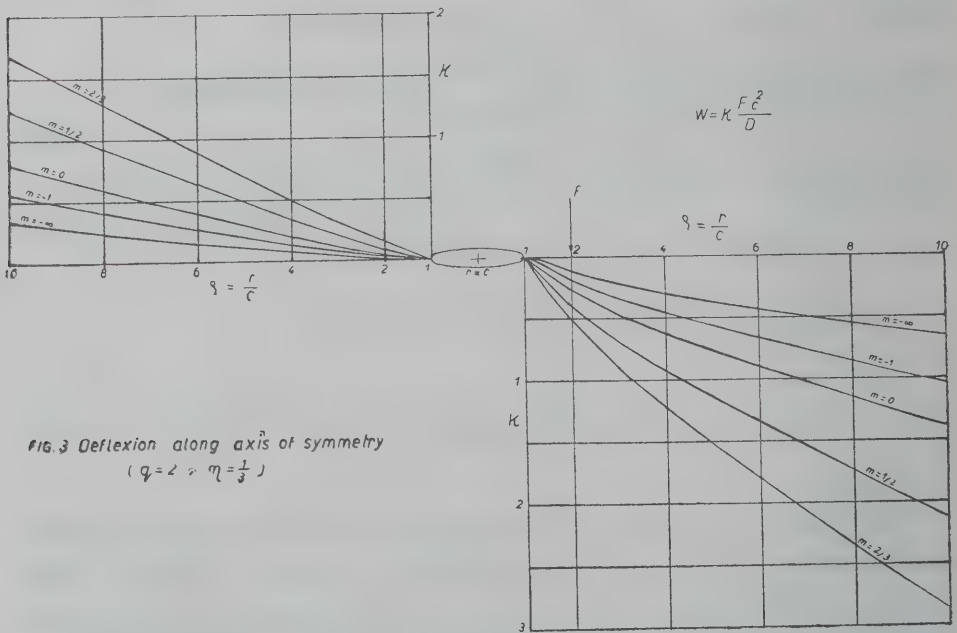


FIG. 3 Deflexion along axis of symmetry
($q=2$, $\nu=\frac{1}{3}$)

The moments and shearing stresses at any point of the plate can now be obtained by substituting from (33) in (30) and the results may be expressed in the form

$$\left. \begin{aligned} M_r &= \delta_1 F , \quad M_\theta = \delta_2 F , \quad M_{r\theta} = \delta_3 F \\ Q_r &= \gamma_1 F/c , \quad V_r = \gamma_2 F/c , \quad Q_\theta = \gamma_3 F/c . \end{aligned} \right\} \quad (37)$$

where, with the notations of Fig. 2.

$$\left. \begin{aligned}
 \delta_1 &= + \frac{1-\eta}{8\pi} \left[m (q^2 - 1) \left(\lambda - \frac{1}{\rho^2} \right) \frac{f'}{R'} \cos \Theta' \right. \\
 &\quad \left. + \cos 2\omega - \frac{R}{R'} \cos (\omega + \omega') \right. \\
 &\quad \left. + \frac{2 \log q}{\lambda \rho^2} \pm \frac{2}{\beta} \log \frac{R}{R'} - m (q^2 - 1) \right. \\
 &\quad \left. \left(\frac{m+1}{\rho^2} - m \lambda \pm \frac{2\lambda}{\beta} \right) C_m (\rho, \theta) \right], \\
 \delta_3 &= \frac{1-\eta}{8\pi} \left[m (q^2 - 1) \left(\lambda - \frac{1}{\rho^2} \right) \frac{f'}{R'} \sin \Theta \right. \\
 &\quad \left. + \sin 2\omega - \frac{R}{R'} \sin (\omega + \omega') - m (q^2 - 1) \right. \\
 &\quad \left. \left(\frac{m+1}{\rho^2} - m \lambda \right) S_m (\rho, \theta) \right],
 \end{aligned} \right\} \quad (38)$$

$$\left. \begin{aligned}
 \gamma_1 &= \frac{1}{2\pi} \left[\frac{c}{R'} \cos \omega' - \frac{c}{R} \cos \omega - \frac{m\lambda}{\rho} (q^2 - 1) \right. \\
 &\quad \left. \left\{ \frac{f'}{R'} \cos \Theta' + m C_m (\rho, \theta) \right\} \right], \\
 \gamma_2 &= \gamma_1 - \frac{\partial}{\partial \theta} \delta_3, \\
 \gamma_3 &= \frac{1}{2\pi} \left[\frac{c}{R'} \sin \omega' - \frac{c}{R} \sin \omega - \frac{m\lambda}{\rho} (q^2 - 1) \right. \\
 &\quad \left. \left\{ \frac{f'}{R'} \sin \Theta' + m S_m (\rho, \theta) \right\} \right],
 \end{aligned} \right\} \quad (39)$$

where $C_m (\rho, \theta)$ is given by (35) and

$$S_m (\rho, \theta) = \sum_{n=1}^{\infty} \frac{(\rho q)^{-n} \sin n \theta}{n - m}. \quad . \quad . \quad . \quad (40)$$

For the boundary values of the stresses we substitute from (33) in (8) and (9). Using the relations (16) it is found that over the boundary C.

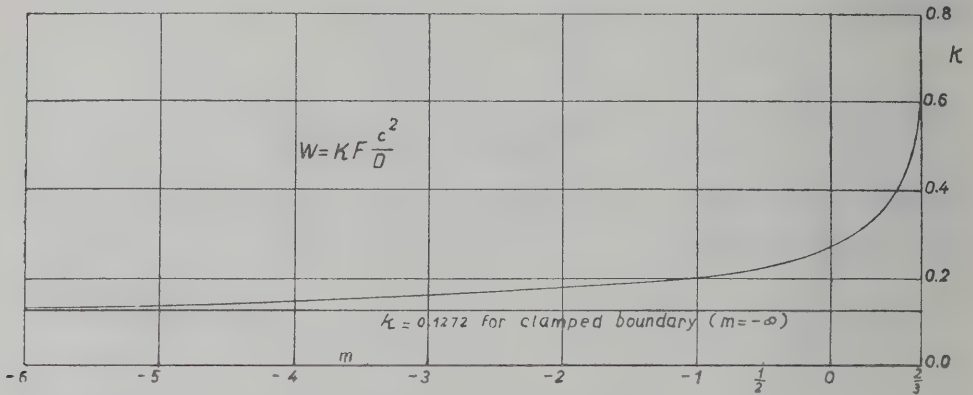


FIG. 4 Deflexion at load point ;

$$\gamma = q = 2, \eta = \frac{1}{3}$$

$$\left. \begin{aligned} \frac{\delta_1}{\lambda + \beta} &= \frac{\delta_2}{\lambda - \beta} = -\frac{1 + \eta}{4 \pi} \left[\frac{\log q}{\lambda} \right. \\ &\quad \left. - m(q^2 - 1) C_m(1, \theta) \right], \\ \frac{\delta_3}{\eta - 1} &= \frac{\gamma_3}{2(1-m)} = \frac{q^2 - 1}{4 \pi} \left[\frac{q \sin \theta}{1 - 2q \cos \theta + q^2} \right. \\ &\quad \left. + m S_m(1, \theta) \right], \\ \gamma_1 &= \frac{m(q^2 - 1)}{2 \pi} \left[\frac{1 - \lambda q \cos \theta}{1 - 2q \cos \theta + q^2} - m \lambda C_m(1, \theta) \right], \\ \gamma_2 &= \gamma_1 + \frac{(1 - \eta)(q^2 - 1)}{4 \pi} \left[\frac{m(q \cos \theta - 1)}{1 - 2q \cos \theta + q^2} \right. \\ &\quad \left. + q \frac{(1 + q^2) \cos \theta - 2q}{(1 - 2q \cos \theta + q^2)^2} + m^2 C_m(1, \theta) \right], \end{aligned} \right\} \quad (41)$$

Over the x-axis, $z = \bar{z} = x$ and if we put $\frac{x}{c} = \epsilon$ we have along the axis of symmetry $\delta_3 = 0 = \gamma_3$ and

$$\left. \begin{aligned} \delta_1 &= + \frac{1-\eta}{8\pi} \left[\frac{m(q^2-1)(1-\epsilon^{-2})}{q\epsilon-1} + \frac{2\log q}{\lambda\epsilon^2} \right. \\ &\quad \left. + \frac{2}{\beta} \log \left| \frac{q^2 - q\epsilon}{1 - q\epsilon} \right| \right. \\ &\quad \left. - m(q^2-1) \left(\frac{m+1}{\epsilon^2} - m\lambda + \frac{2\lambda}{\beta} \right) \sum_{n=1}^{\infty} \frac{(q\epsilon)^{-n}}{n-m} \right], \\ \gamma_1 &= \frac{m(q^2-1)}{2\pi\epsilon} \left[\frac{\epsilon - \lambda q}{(q-\epsilon)(q\epsilon-1)} \right. \\ &\quad \left. - m\lambda \sum_{n=1}^{\infty} \frac{(q\epsilon)^{-n}}{n-m} \right]. \end{aligned} \right\} \quad (42)$$

In the case of a clamped inner boundary the stresses are obtained from the previous results as the limiting values when $\lambda \rightarrow 1$ and $m \rightarrow -\infty$. However, this case may be treated independently since Equations (25), (26) and (31) give in this case

$$G = \log q, \quad H = 0, \quad A = 1 - q^2 + 2 \log q, \quad (43)$$

so that using (23), (30) and (43) we find that over a clamped inner circular boundary

$$\left. \begin{aligned} \delta_1 &= \frac{\delta_2}{\eta} = \frac{1}{2\pi} \left[\frac{(1-q\cos\theta)(q^2-1)}{1-2q\cos\theta+q^2} - \log q \right], \quad \delta_3 = 0, \\ \frac{\gamma_1}{q\cos\theta-1} &= \frac{\gamma_3}{q\sin\theta} = \frac{1}{2\pi} \left[\frac{1-q^2}{1-2q\cos\theta+q^2} \right]^2, \quad \gamma_2 = \gamma_1. \end{aligned} \right\} \quad (44)$$

In such a case we get along the x-axis

$$\left. \begin{aligned} \delta_1 &= + \frac{1-\eta}{8\pi} \left[\frac{2 \log q}{\varepsilon^2} - \frac{(q^2-1)(1-2q\varepsilon+\varepsilon^2)}{\varepsilon^2(q\varepsilon-1)^2} \right] \\ &\pm \frac{2}{\beta} \left\{ \frac{q^2-1}{q\varepsilon-1} + \log \left| \frac{q^2-q\varepsilon}{1-q\varepsilon} \right| \right\}, \\ \gamma_1 &= \frac{(1-q^2)^2}{2\pi(q-\varepsilon)(q\varepsilon-1)^2}. \end{aligned} \right\} \quad (45)$$

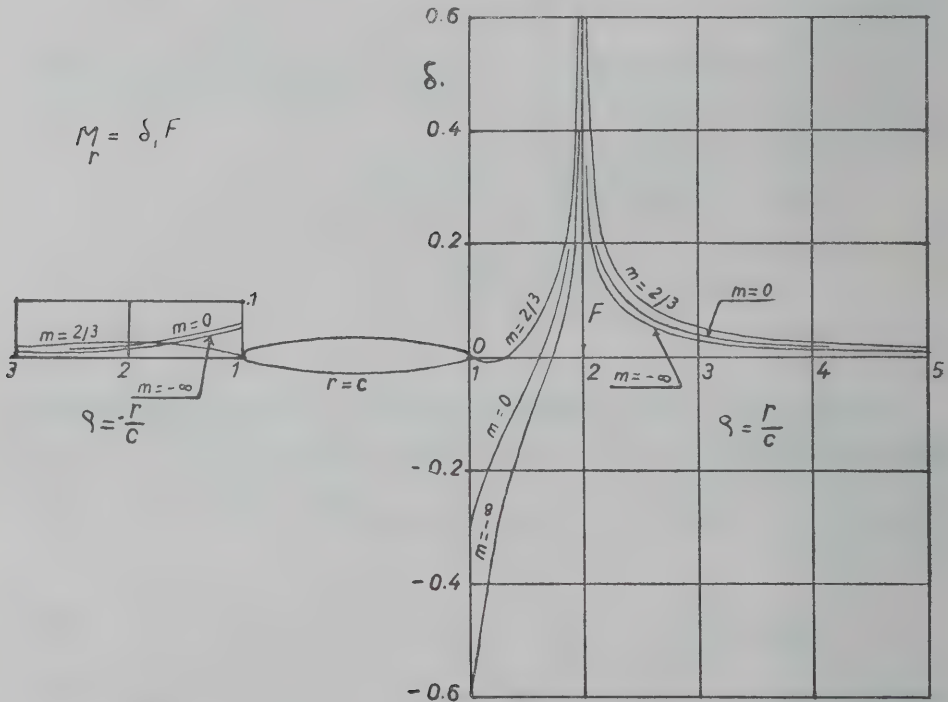


FIG.5 Bending moment M_r along axis of symmetry; $q=2$, $\eta_l=\frac{1}{3}$

Calculations were carried out for different values of the restraining parameter m . In each case the stresses and deflection were computed for various points along the axis of symmetry of the plate and on the circular boundary. Figs 5, 6 and 7 show some of the results

obtained for the bending moments and shearing force along the axis of symmetry. In Figs. 8, 9 and 10 we plot typical curves for the boundary values of the radial bending moment M_r , shear stress Q_r and vertical reaction V_r .

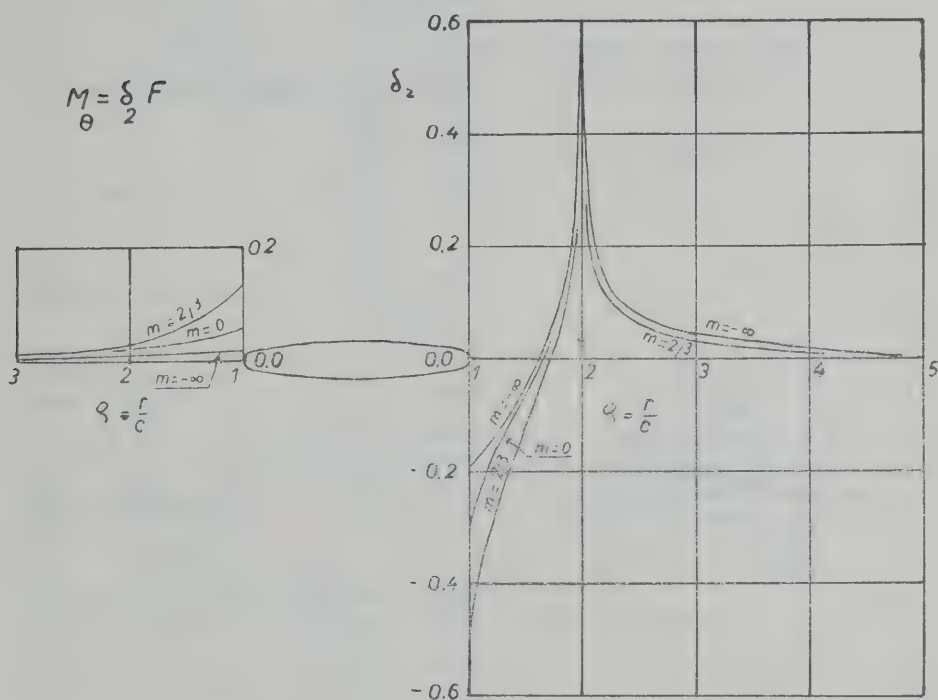


FIG. 6 Bending moment M_θ along axis of symmetry; $q=2$, $\eta=\frac{1}{3}$

It is of interest to notice that the effect of the elastic restraint of the boundary on the stresses is limited to the neighbourhood of the boundary and that the stresses decrease rapidly as we go further from the load point. The present solution for the infinite plate may be then adopted for a finite plate elastically restrained over an inner circular boundary with outer edge free provided that the concentrated force is at a reasonable distance from the outer edge.

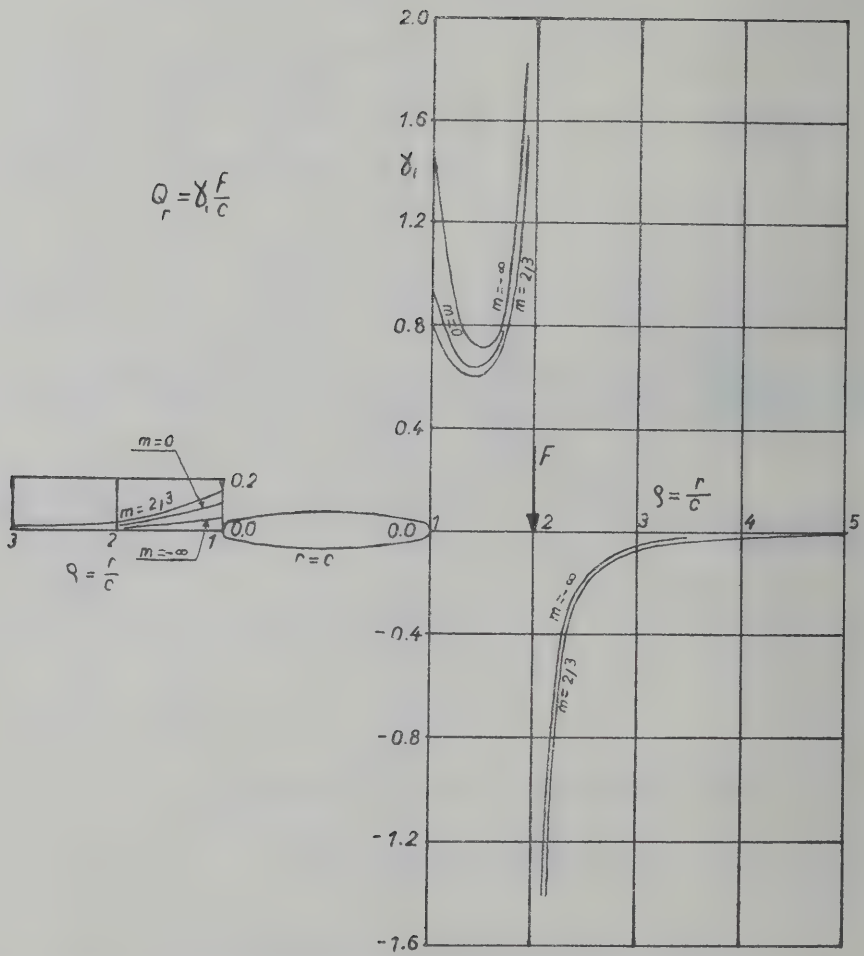


FIG.7 Shear stress Q_r along axis of symmetry ; $q=2$, $\eta_1 = \frac{1}{3}$

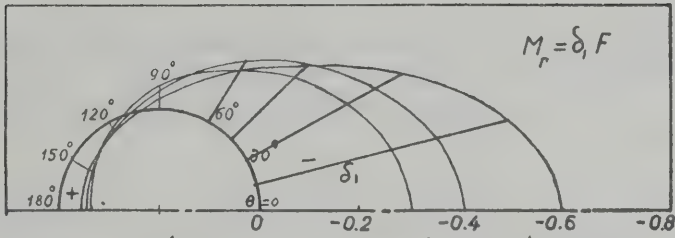


FIG.8 Bending Moment at circular boundary

$$q=2, \eta=\frac{1}{3}$$

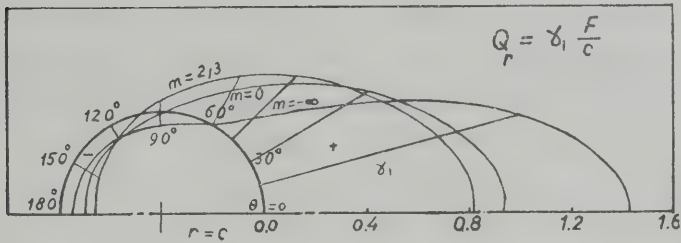


FIG.9 Shear stress at circular boundary

$$q=2, \eta=\frac{1}{3}$$

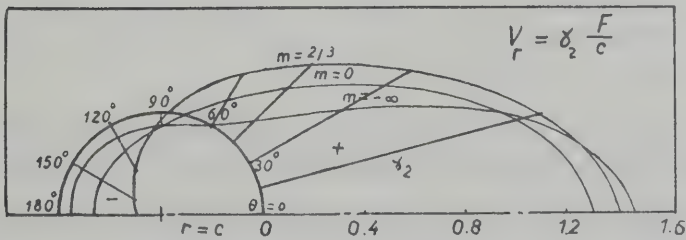


FIG.10 Vertical reaction at circular boundary

$$(q=2; \eta=\frac{1}{3})$$

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MECHANICAL ENGINEERING

RIPPLED BEARINGS FOR HIGH SPEED JOURNALS

BY

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This paper presents an analytical study of the performance of a complete journal bearing with sinusoidally undulating bearing surface (referred to as rippled bearing). The study is herein confined to light-load high-speed applications, in which concentric running of journal may plausibly be assumed. Both cases of full-and partial-width films in the bearing are considered.

The rippled bearing is compared with the elliptical (2-lobe) bearing on basis of same size, mean clearance and minimum film thickness, and is shown to have invariably lower friction values. It is also found to give increased end flow when the bearing has three or more ripples round its periphery. This increased flow together with reduced friction would lead to lower operating temperature, and consequently higher shaft stability.

These favourable features of the rippled bearing would recommend it for use with journals of increasingly high rotative speeds.

INTRODUCTION

Designers of modern machinery are continually striving for higher rotative speeds, a trend which brings in new problems, among which are increased power loss, excessive bearing temperature and greater tendency to shaft whirl due to oil-film action in the bearing.

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For these conditions of operation, bearings of non-circular profiles have become more and more desirable, e.g. elliptical, 3- and 4-lobe bearings (5)(¹). These profiles are so chosen as to form convergent passages which, with eccentric running of journal, would set up additional loads in the bearing which assist in enhancing shaft stability.

In the present paper, a new type of non-circular profile is investigated. This profile is composed of sinusoidal undulations covering the whole periphery of the bearing. Two cases of film extent are considered, viz. full-width and partial-width films in divergent passages, with full-width film prevailing in convergent passages.

Friction values and end flow are obtained, and their effect on shaft stability is discussed. The results are further compared with those of the elliptical bearing.

NOTATION

- a_o : Amplitude of bearing surface undulation.
- B : Axial bearing width.
- B' : Axial film width in divergent passage (or combined width of film streamlets).
- c : Mean value of radial bearing clearance.
- c_o : Minimum value of radial bearing clearance in an elliptical bearing (see Fig. 4).
- F_j : Total frictional force on journal.
- h : Fluid film thickness.
- $h_{\min}$: Minimum value of fluid film thickness.
- k : Ratio of frictional force in a non-circular cylindrical bearing to that of a circular cylindrical bearing of same diameter, axial width and mean radial clearance c .

(¹) An alphabetical list of references is given at the end of the paper.

- m : Amplitude ratio of bearing surface undulation = a_o / c .
 m_e : = $s / (c_o + s)$.
 n_o : Number of ripples (surface sinusoidal undulations) round bearing periphery.
 N : Rotational journal speed.
 Q : End flow of lubricant.
 r : Radius of journal.
 R_l : Radius of lobe (see Fig. 4).
 s : Difference between maximum and minimum radial clearances in an elliptical bearing = $[R_l - (r + c_o)]$; see Fig. 4.
 U : Circumferential velocity of journal surface = ωr .
 θ : Angular position of any point in the fluid measured in direction of journal rotation.
 λ : Coefficient of viscosity of lubricant.
 τ : Shear stress in fluid in circumferential direction.
 ω : Angular speed of journal rotation.

ANALYTICAL TREATMENT

It is assumed that the journal has a perfect circular profile, the bearing a sinusoidally undulating profile, that the flow is streamlined and that the fluid viscosity is uniform throughout the film.

The bearing profile may conveniently be represented by $-a_o \cos (n_o \theta)$. The film thickness will then be given by:

$$h = [c + a_o \cos (n_o \theta)] \quad . \quad . \quad . \quad . \quad . \quad (1)$$

In high speed machinery in which journal bearings are expected to operate under conditions of light loads and high speeds, the resultant force due to fluid pressure would approximate to zero and the shaft would assume a concentric running position.

With integer values of n_o equal to or higher than two, Fig. 1, concave as well as convex parts of the surface are symmetrically disposed with respect to the bearing centre; consequently no resultant

Due to the incapability of fluids to sustain tensile stresses, the lubricating film would tend to cavitate where negative pressure be liable to occur.

There is experimental evidence of narrowing down of the axial extent of the film in a divergent passage, and of breaking up of the outlet film into streamlets (2); no side leakage therein would be expected to take place.

In this analysis two cases are considered, viz.:

- (i) full-width film, i.e. full-width film in both convergent and divergent passages.
- (ii) partial-width film, i.e. full-width film in convergent passages and partial-width film in divergent passages.

The results are compared with those of the elliptical (2-lobe) bearing (5).

(i) Full-Width Film:

Force of friction on journal

$$\begin{aligned} F_j &= B \int_0^{2\pi} \tau \cdot r \, d\theta = \lambda U r B \int_0^{2\pi} \frac{d\theta}{h} \\ &= \lambda U r B \int_0^{2\pi} \frac{d\theta}{[c + a_o \cos(n_o \theta)]} \end{aligned}$$

Introducing $a_o = m c$, $0 < m < 1$,

$$\begin{aligned} \therefore F_j &= \frac{\lambda U r B}{c} \int_0^{2\pi} \frac{d\theta}{[1 + m \cos(n_o \theta)]} \\ &= \frac{\lambda U r B}{c n_o} \left[\frac{1}{\sqrt{1 - m^2}} \sin^{-1} \left\{ \frac{\sqrt{1 - m^2} \sin(n_o \theta)}{[1 + m \cos(n_o \theta)]} \right\} \right]_0^{2\pi} \\ &= \frac{[(2\pi r B) \frac{\lambda U}{c}]}{\sqrt{1 - m^2}} \dots \dots \dots (3) \end{aligned}$$

It is evident that the force of friction on the journal is independent of the number of ripples round the bearing periphery. Convergent and divergent parts of the film contribute equally to the frictional force.

For $m = 0$,

$$[F_j]_{m=0} = \left[(2 \pi r B) \lambda \frac{U}{c} \right] \quad (4)$$

first obtained by Petroff in 1883 (4), and referred to in the text of this paper as "Petroff's Frictional Force".

∴ The ratio of the frictional force on the journal to Petroff's frictional force will be given by :

$$k = \frac{1}{\sqrt{1 - m^2}} \quad (5)$$

(ii) *Partial-Width Film:*

It is assumed that full-width film exists in convergent passages, while only partial-width film prevails in divergent passages, Fig. 2. In these latter passages, the axial extent of the film is determined from requirements of volume continuity (7), viz. :

$$B' h = \text{constant} = B h_{\min} \quad (6)$$

$$\text{or } B' = \frac{B (1 - m)}{(1 + m \cos n_0 \theta)} \quad (7)$$

For convergent passages,

$$\begin{aligned} \text{Force of friction} &= \frac{\lambda U r B}{c} n_0 \int_0^{\pi/n_0} \frac{d\theta}{(1 + m \cos n_0 \theta)} \\ &= \frac{[(\pi r B) \lambda \frac{U}{c}]}{\sqrt{1 - m^2}} \quad (8) \end{aligned}$$

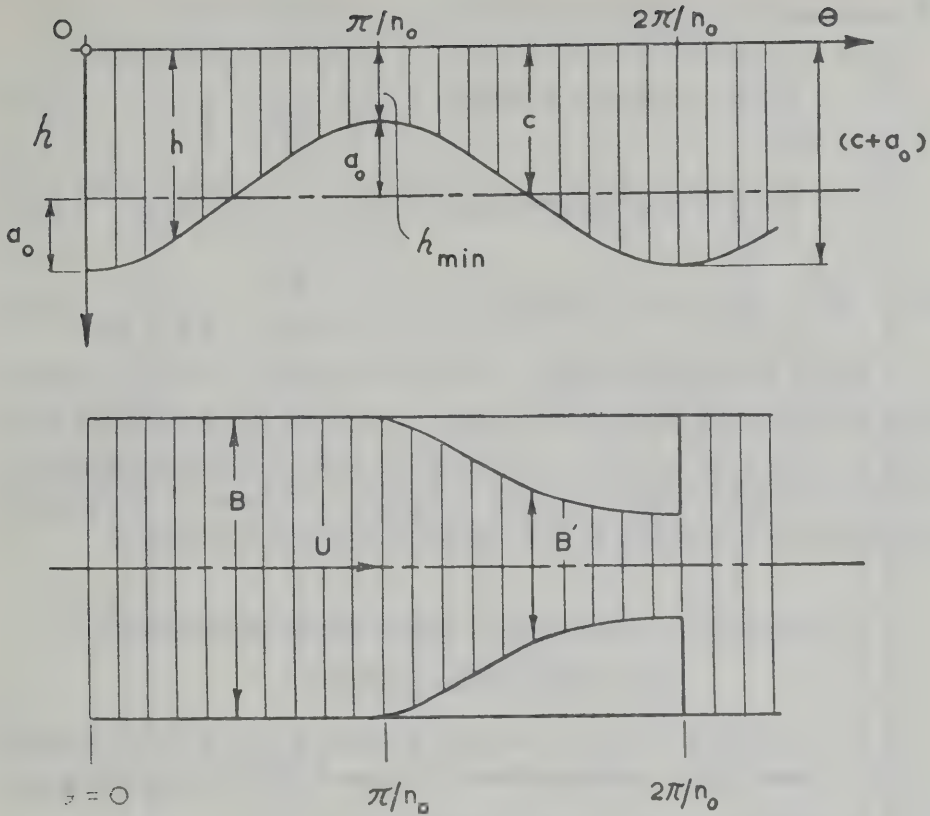


FIG. 2.—Geometry of Partial-Width Film in a Rippled Bearing.

For divergent passages,

$$\text{Force of friction} = \frac{\lambda U r}{c} n_0 \pi/n_0 \int_0^{2\pi/n_0} \frac{B' d\theta}{(1 + m \cos n_0 \theta)}$$

Substituting for B' from equation (7),

$$\begin{aligned} \dots \text{Force of friction} &= \frac{\lambda U r B (1 - m)}{c} \pi/n_0 \int_0^{2\pi/n_0} \frac{d n_0 \theta}{(1 + m \cos n_0 \theta)^2} \\ &= \frac{\lambda U r B (1 - m)}{c (1 - m^2)} \left[\frac{1}{\sqrt{1 - m^2}} \sin^{-1} \left\{ \frac{\sqrt{1 - m^2} \sin n_0 \theta}{(1 + m \cos n_0 \theta)} \right\} \right. \\ &\quad \left. - \frac{m \sin n_0 \theta}{(1 + m \cos n_0 \theta)} \right]_{\pi/n_0}^{2\pi/n_0} = \frac{[(\pi r B) \lambda \frac{U}{c}]}{(1 + m) \sqrt{1 - m^2}} \end{aligned} \quad (9)$$

From equations (4), (8) and (9), it can be concluded that :

$$k \text{ for convergent passage/s} = \frac{1}{\sqrt{1 - m^2}} \quad (10)$$

$$k \text{ for divergent passage/s} = \frac{1}{(1 + m)} \cdot \frac{1}{\sqrt{1 - m^2}} \quad (11)$$

$$\text{and } k \text{ for the whole bearing} = \frac{(2 + m)}{2(1 + m)} \cdot \frac{1}{\sqrt{1 - m^2}} \quad (12)$$

Fig. 3 shows the variation of k with m for the two cases of full- and partial-width films. The latter case gives, for amplitude ratio values of up to about 0.577, lower friction than that of a plain bearing of same size and mean clearance, a feature in favour of the rippled bearing over its amplitude-ratio range of practical importance.

COMPARISON OF FRICTION VALUES BETWEEN RIPPLED AND ELLIPTICAL BEARINGS

The so-called elliptical bearing is made up of two cylindrical halves (lobes) with their centres displaced from bearing centre, Fig. 4.

From bearing geometry, it can be shown that :

$$(h + r) = R_1 \cos \alpha + s \cos \theta$$

With α being very small, the film thickness in an elliptical bearing may be given by :

$$h = (c_o + s) + s \cos \theta \quad (13)$$

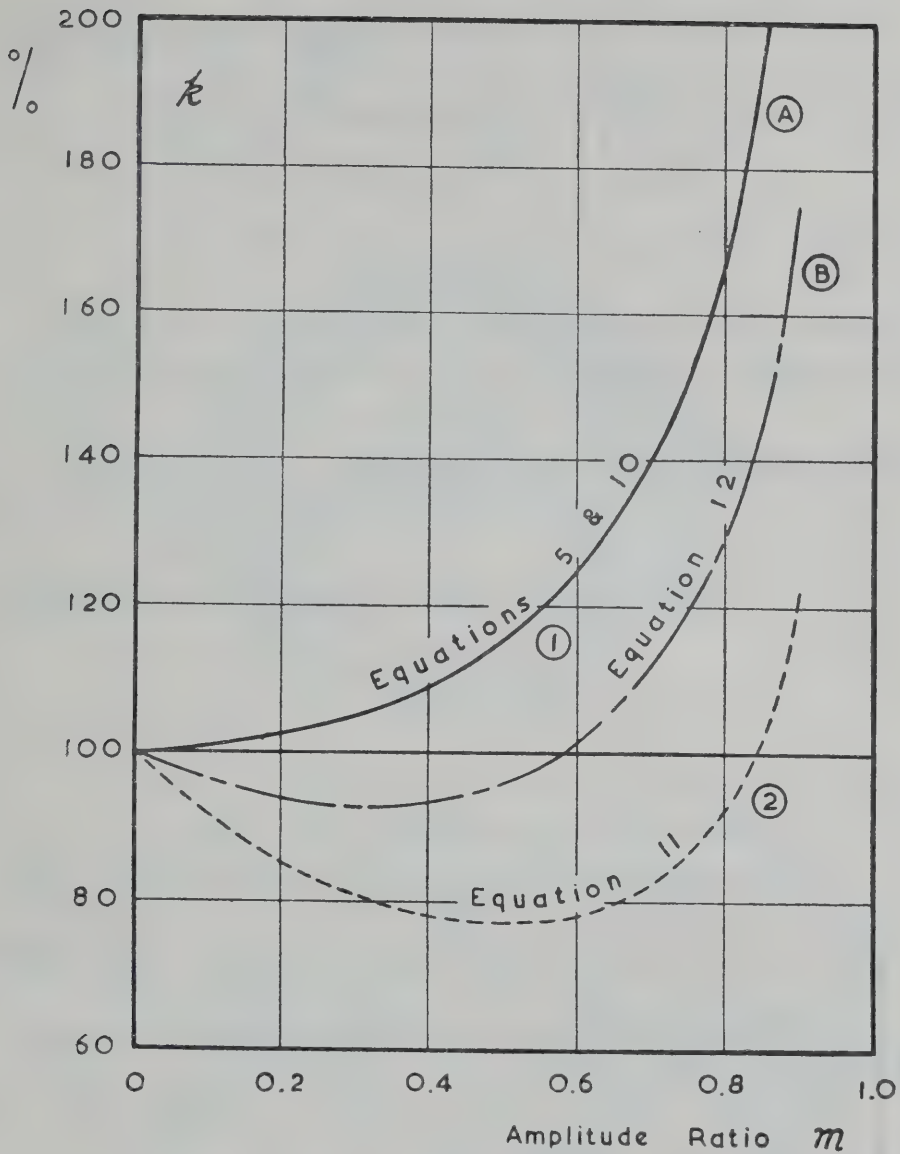
where θ varies between $\pi/2$ and $3\pi/2$ for each bearing half.

Considering frictional tractions in one bearing half,

Force of friction in convergent part of film

$$= \lambda U r B \int_{\pi/2}^{\pi} \frac{d\theta}{(c_o + s) + s \cos \theta}$$

Introducing $m_e = s / (c_o + s)$,



- ① Convergent passage/s with full-width film
- ② Divergent passage/s with partial-width film
- (A) Full-width film in bearing
- (B) Partial-width film in bearing

FIG. 3.- k - m Relationship for the Rippled Bearing.

∴ Mean radial bearing clearance

$$c = \frac{1}{\pi} \int_{\pi/2}^{3\pi/2} h \, d\theta = (c_o + s) - \frac{2}{\pi} s \quad (15)$$

∴ k for convergent passage in one bearing half

$$= \frac{2}{\pi} \left(1 - \frac{2}{\pi} m_e \right) \frac{1}{\sqrt{1 - m_e^2}} \left[\pi - \sin^{-1} \sqrt{1 - m_e^2} \right] \quad (16)$$

Should the divergent passage be completely full of lubricant, k will be the same for both convergent and divergent passages, so will be k for the whole bearing (case of full-width film).

If the film extent in the axial direction, for the divergent passage, is, however, governed by the relationship :

$$B h_{\min} = B' h ,$$

force of friction in divergent passage will be given by :

$$\left. \begin{aligned} & \frac{\lambda U r B c_o}{(c_o + s)^2} \pi \int_{\pi/2}^{3\pi/2} \frac{d\theta}{(1 + m_e \cos \theta)^2} \\ &= \frac{\lambda U r B c_o}{(c_o + s)^2} \frac{1}{(1 - m_e^2)} \left[m_e + \frac{1}{\sqrt{1 - m_e^2}} \right. \\ & \quad \left. \left(\pi - \sin^{-1} \sqrt{1 - m_e^2} \right) \right] \end{aligned} \right\} \quad (17)$$

$$\text{and } k = \frac{2}{\pi} \left(1 - \frac{2}{\pi} m_e \right) \frac{1}{(1 + m_e)} \left[m_e + \frac{1}{\sqrt{1 - m_e^2}} \right. \left. \left(\pi - \sin^{-1} \sqrt{1 - m_e^2} \right) \right] \quad (18)$$

∴ k for the whole bearing with partial-width film is given by :

$$\left. \begin{aligned} k &= \frac{1}{\pi} \left(1 - \frac{2}{\pi} m_e \right) \left\{ \frac{m_e}{(1 + m_e)} + \frac{1}{\sqrt{1 - m_e^2}} \right. \\ & \quad \left. \frac{(2 + m_e)}{(1 + m_e)} \left(\pi - \sin^{-1} \sqrt{1 - m_e^2} \right) \right\} \end{aligned} \right\} \quad (19)$$

Friction values for the rippled bearing, Fig. 3, and for the elliptical bearing, Fig. 5, are compared on basis of equal values of mean radial bearing clearance c and minimum film thickness $h_{\min}$.

$$\left. \begin{array}{l} \text{i.e.} \quad (1 - m) c = c_0 \\ \text{and} \quad c = \left(\frac{1}{m_e} - \frac{2}{\pi} \right) s \end{array} \right\} \quad \cdot \quad \cdot \quad \cdot \quad (20)$$

$$\therefore \quad m = 1 - \frac{(1 - m_e)}{\left(1 - \frac{2}{\pi} m_e \right)} \quad \cdot \quad \cdot \quad \cdot \quad (21)$$

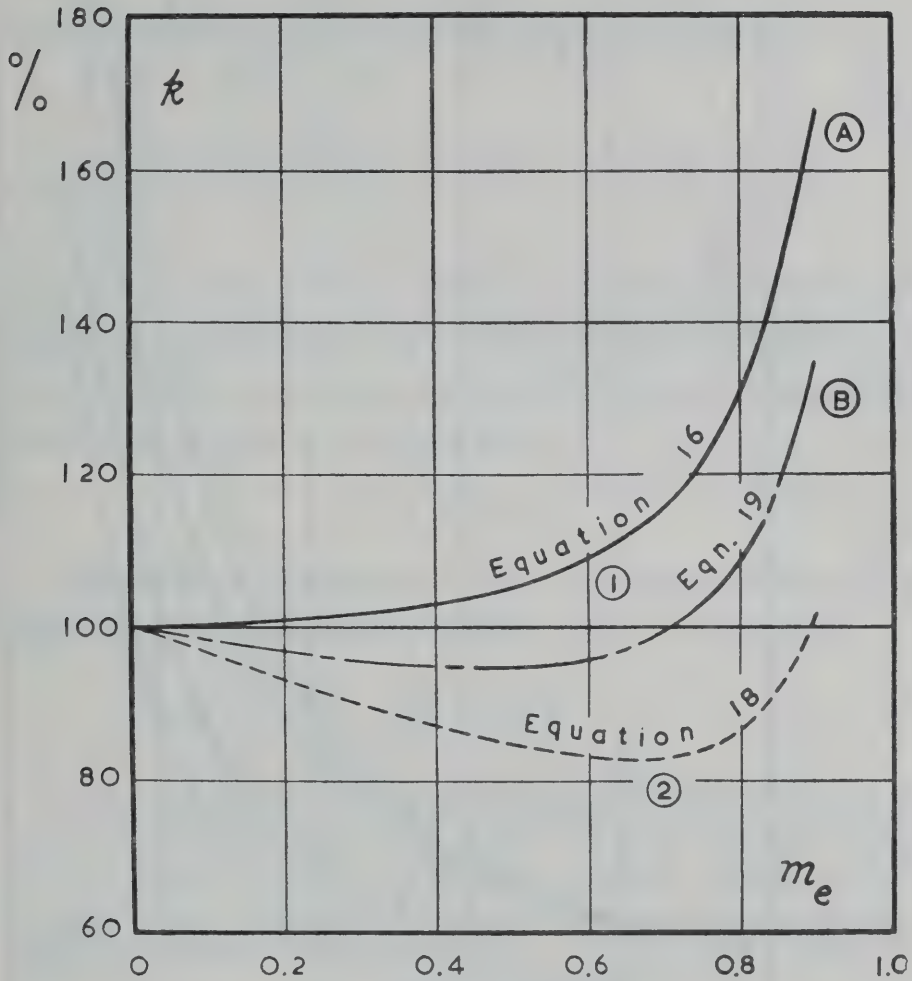
The rippled bearing shows lower friction than the elliptical bearing, Fig. 6, for both cases of full-and partial-width films, the difference attaining some 10% at the higher values of m . Lower friction would lead to lower bearing temperature and consequently higher shaft stability (6).

It should be noted that an increase in the number of ripples (with n_0 as integer) does not affect friction, in contradistinction to an increase in the number of lobes (on a multi-lobe bearing) which results in higher friction (5). This is why relevant comparison is only made, herein, with the elliptical (2-lobe) bearing.

FRACTIONAL FILM AREA UNDER SHEAR

The fractional film area under shear (in relation to the whole surface area) for the rippled bearing with partial-width film is given by:

$$\left. \begin{aligned} & \frac{1}{2} + \frac{n_0 h_{\min}}{2 \pi} \int_{\pi/n_0}^{2\pi/n_0} \frac{d \theta}{h} \\ &= \frac{1}{2} + \frac{(1 - m)}{2 \pi} \int_{\pi}^{2\pi} \frac{d (n_0 \theta)}{[1 + m \cos (n_0 \theta)]} \\ &= \frac{1}{2} + \frac{(1 - m)}{2 \sqrt{1 - m^2}} \end{aligned} \right\} \quad (22)$$



- ① Convergent passage/s with full-width film
- ② Divergent passage/s with partial-width film
- (A) Full-width film in bearing
- (B) Partial-width film in bearing

FIG. 5.— k - m_e Relationship for the Elliptical Bearing.

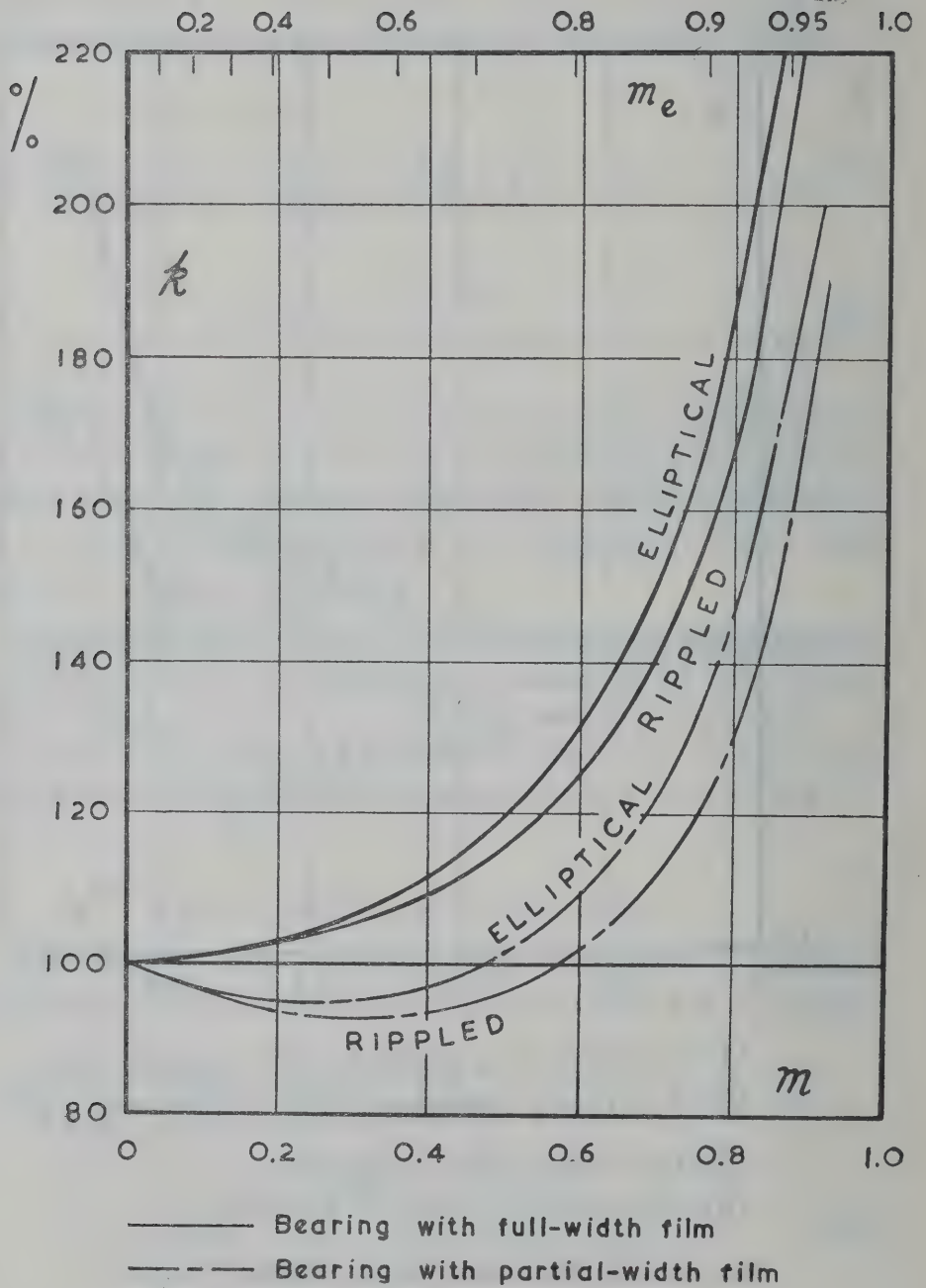


FIG. 6.—Comparison of Friction Values between Rippled and Elliptical Bearings.

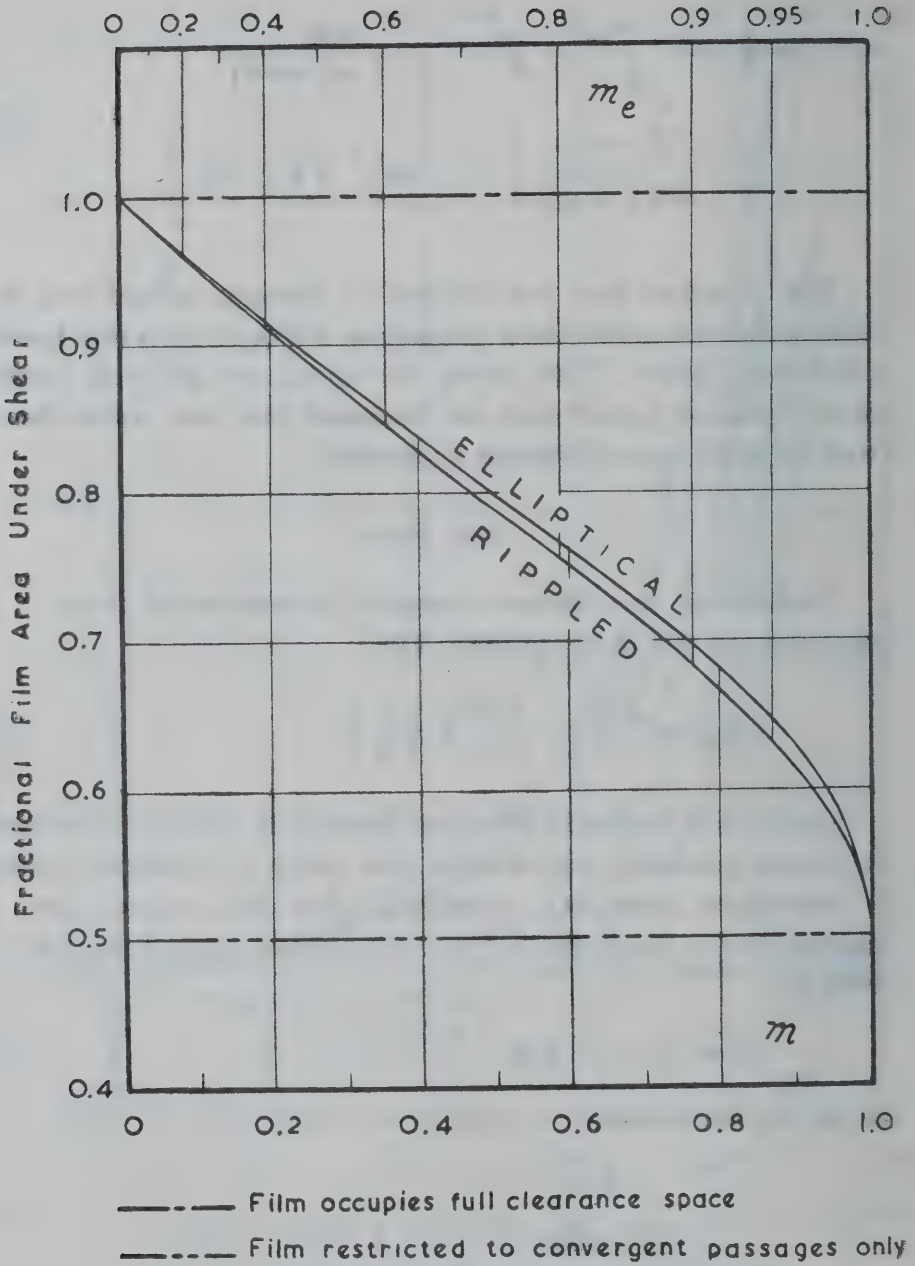


FIG. 7.—Fractional Film Area under Shear.

$$\therefore \frac{(Q) \text{ rippled}}{(Q) \text{ elliptical}} = n_o \cdot m \left(\frac{1}{m_e} - \frac{2}{\pi} \right) \quad (27 a)$$

which, for same values of c and $h_{\min}$ in the two bearings,

$$= n_o \left(1 - \frac{2}{\pi} \right) \quad (27 b)$$

This ratio reaches unity when $n_o \approx 2.75$. It can be concluded, therefore, that end flow in the rippled bearing exceeds that of the elliptical bearing provided that n_o is at least 3. As shown by equation (27 b) and in Fig. 8, the superiority of the rippled bearing with regard to end flow increases linearly with the number of ripples on the bearing surface. This would provide some means of reducing the operating bearing temperature, a feature much desired for high speed journals.

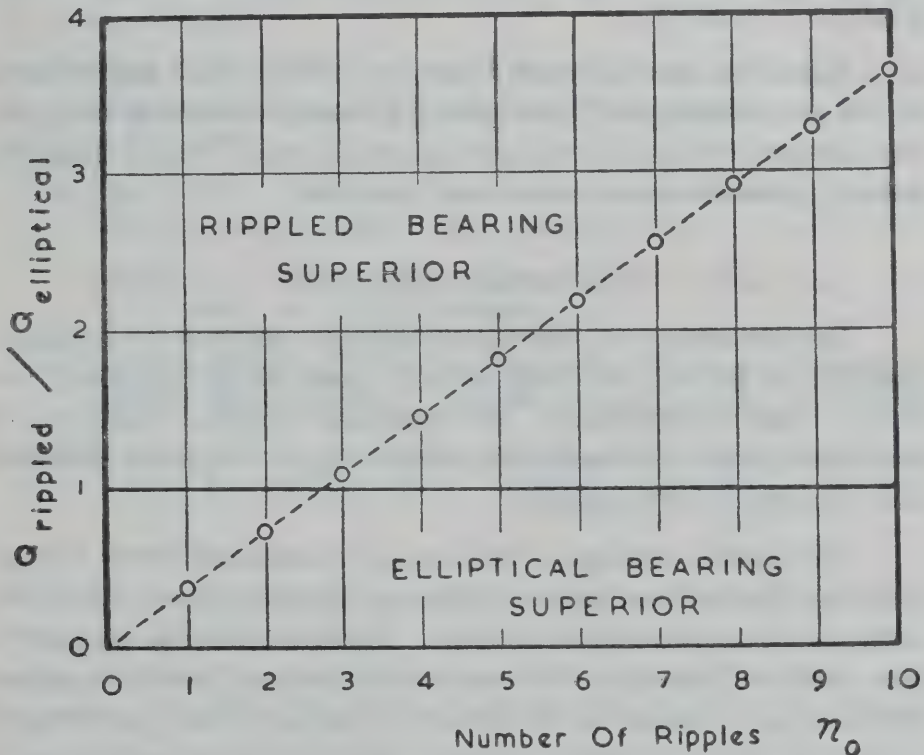


FIG. 8.—Comparison of end Flow in Rippled and Elliptical Bearings.

JOURNAL STABILITY

The convergent parts of the rippled bearing simulate, in action, those of the multi-pad journal bearing. With eventual eccentric running of journal, fluid film pressures will develop in the individual convergent passages, and relevant forces set up in the bearing (1) will strive for bringing the journal back to concentric running. In this way, surface ripples play their significant role in promoting journal stability. The fact that the tilting pad journal bearing is incapable of exciting or sustaining journal whirl has already been established in theory and by experiment (3).

Further, the foregoing analysis shows that the rippled bearing gives lower friction, also increased end flow (for $n_0 \geq 3$) in comparison with the elliptical bearing. With these features, reduced bearing temperature would result, and consequently greater shaft stability would be expected (6).

Should the journal be acted upon by a small load, it may be made to run concentrically with the bearing by using a fractional value for the number of ripples on the bearing surface, which ripples should be suitably oriented with respect to the load line.

SUMMARY AND CONCLUSION

The performance of the rippled bearing is obtained for concentric running of journal, two cases of film extent being considered, viz. full-and partial-width films. The results are compared with those of the elliptical (2-lobe) bearing on basis of equal size, mean clearance and minimum film thickness.

The rippled bearing is shown to give invariably lower friction than the elliptical bearing, the difference attaining some 10% at the higher values of the amplitude ratio. Moreover, while an increase in the number of lobes (e.g. 3-lobe and 4-lobe bearings) results in higher friction (5), an increase in the number of ripples (With n_0 as integer) produces no change in friction. It can be concluded, therefore, that, in this respect, the rippled bearing is superior to the multi-lobe bearing.

The rippled bearing is also found to have higher end flow than that of the elliptical bearing provided the number of ripples is at least three.

These performance characteristics of the rippled bearing result in relatively reduced power loss, hence higher bearing efficiency, also lower operating temperature and greater journal stability. Such qualifications may well recommend this type of bearing as a satisfactory solution to the problem of bearing design for high speed journals.

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INVESTIGATION OF HEAT TRANSFER TO AIR IN RADIAL FLOW

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ABSTRACT

In the work described here air was flowing between two parallel plates in an outward radial direction. One of the plates was electrically heated, while the other one was insulated against external radiation. Eight different gaps between the plates were tested, and the average coefficients of heat transfer were obtained.

The effect of the gap on the heat transfer was obtained for both laminar and turbulent flows. All the data of the different gaps were brought together by introducing equations containing the Nusselt and Reynolds numbers and the gap-to-length ratio.

NOMENCLATURE

The symbols mentioned in the text have the following meanings :

b = gap between the two plates, (inch).

h = coefficient of heat transfer, $\text{Btu.}/(\text{ft})^2 (\text{hr}) (\text{deg. F})$.

H = quantity of heat transfered by convection, $\text{Btu.}/\text{hour}$.

k = thermal conductivity of air, $\text{Btu.}/(\text{ft}) (\text{hr}) (\text{deg. F})$.

L = legth of the heating surface, $= r_2 - r_1$.

Q = rate of air discharge, pounds per hour.

r_1 = inner radius of the heat transfer surface.

r_2 = outer radius of heat transfer surface.

S = area of heat transfer surface. squ. ft.

t_w = wall temperature, deg. F.

t_a = bulk temperature of air at the working section, deg. F.

U_m = mean effective velocity of the air stream, feet/sec.

U_1 = air stream velocity at radius r_1 , feet/sec.

μ = viscosity of air, pounds/(ft.) (hr.).

R_e = Reynolds number.

N_u = Nusselt number.

INTRODUCTION

The heat transfer from a surface to a fluid flowing past it depends, to a great extent, on the flow conditions. Different cases have been studied by different observers. But the case of variable main stream velocity has not had much study until the present time.

In the present work (see Fig. 1) air was sucked from the room by means of an air blower (1) and blown into the main pipe (2). At the end of the pipe air passes between the sides of the working section (3 and 4). The side (3) is the heating unit, while (4) is an unheated disc.

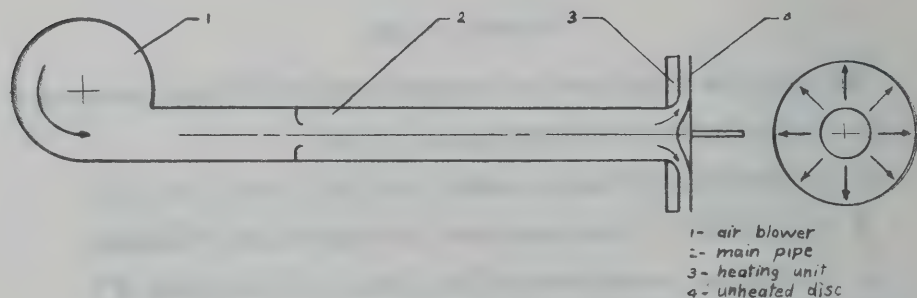


FIG. 1.—Diagram of Apparatus.

This work was carried out for the following purposes :

1. To determine the relation between the average coefficient of heat transfer and the mean effective speed of the flowing air stream.
2. To determine the effect of the gap between the two plates on the coefficient of heat transfer.

APPARATUS

The main pipe was a horizontal circular 6 inch one. A brass nozzle was placed inside the pipe to measure the rate of air discharge.

The heating surface was a circular brass plate of diameters 9 and 16 inches. Two electric heaters were used; the main heater to supply heat to the heating surface, and the guard heater to prevent the heat flow at the rear side of the main heater. Heat flow from the sides of the heating surface was much reduced by surrounding it by asbestos rings. Twenty copper-constantan thermocouples were distributed into the heating unit in order to have full knowledge and control of the temperatures everywhere.

The unheated disc was an 18 inches circular plate, mounted on a steel shaft. To determine its temperature 7 thermocouples were used. Special holes were made to give the static pressures at radii $4\frac{5}{8}$ and $7\frac{7}{8}$ inches, and others to allow for a small total head tube to move between the two plates to measure the velocity distribution between them at the same radii. Two other holes were made for two thermocouples to measure the temperature of the flowing air at radii $4\frac{1}{2}$ and 8 inches. Precautions were made to prevent heat conduction.

A special micrometer and small total head tube arrangement was used to measure the velocity distribution between the two plates. The inner and outer radii of the tube were 0.014 and 0.026 inches. The arrangement was carried by the body of the unheated disc.

The apparatus and the instruments were carried on a bench, the heater being mounted in a vertical plane. A photograph of the final set-up is shown in figure 2.

Special calibration was done to the thermocouples. The nozzle was calibrated in situ. The coefficient of the small total head tube was determined by direct comparison with a standard Pitot tube.

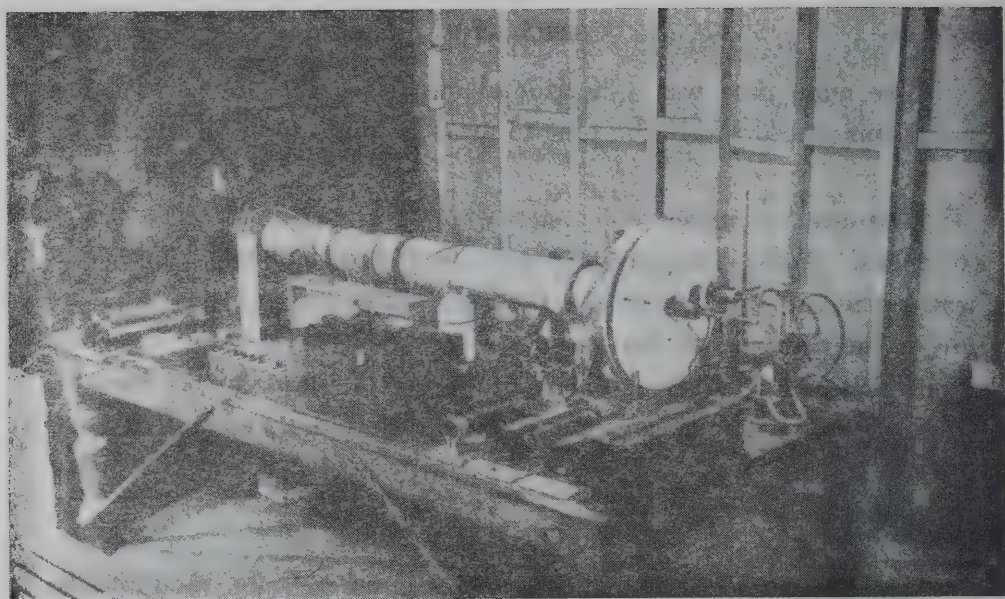


FIG. 2.—Photograph of Apparatus.

PROCEDURE

With a certain gap between the plates, the speed of the air blower was adjusted to give any required mean effective speed of the air stream in the working section. Reasonable electrical inputs were given to the main and guard heaters. Readings of the temperatures were taken every 30 minutes till a thermal steady state was reached. The average coefficient of heat transfer was then calculated.

This experiment was repeated with other air discharges, till the behaviour of the heat transfer for that particular gap was completely known. The gap was then changed to another determined value, and the previous procedure repeated.

In this research eight different gaps were tested. The sum of 126 experiments was carried out. The average time required for every experiment was about 3 hours.

In 14 of these experiments the velocity distribution between the plates was measured. This process always started after reaching the thermal steady state.

METHOD OF CALCULATION

The amount of heat lost by radiation and conduction from the surface and sides of the heating plate was calculated from the temperatures of the hot surface, the unheated disc and the insulating asbestos rings. The amount of heat transferred by convection was then calculated. The average coefficient of heat transfer is given by

$$h = \frac{H}{S. (t_w - t_a)} \text{ B t u. / (ft.)}^2 \text{ (hr.) (deg. F)} \quad (1)$$

The rate of air discharge was directly determined from the calibration curve of the nozzle. The point values of the air stream velocity were calculated from the velocity head and the density of the air. The effective mean air stream velocity in the working section was calculated from the equation.

$$U_m = \frac{2 U_1 r_1}{r_2 + r_1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

In the present work the Reynolds number was put in the form

$$R_e = \frac{Q}{r_1 \mu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The formula chosen to calculate the Nusselt number is

$$N_u = \frac{h b}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The physical properties of the air were always evaluated at the mean bulk temperature of the air stream.

RESULTS

In both the laminar and turbulent ranges the results can be represented by equations (5) and (6) and table I as follows:

$$h = K \cdot U_m^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$N_u = A \cdot R_e^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

TABLE 1
VALUES OF THE CONSTANTS K, A, AND n IN EQUATIONS 5 AND 6

Gap (inches)	Laminar flow			Turbulent flow		
	K	A	n	K	A	n
1/16	2.780	0.130	0.36	—	—	—
1/8	2.370	0.172	0.36	0.3061	0.00045	0.977
3/16	2.220	0.209	0.36	0.851	0.0050	0.773
1/4	2.080	0.236	0.36	1.201	0.0128	0.698
3/8	—	—	—	1.680	0.0341	0.627
1/2	—	—	—	1.986	0.0507	0.611
3/4	—	—	—	2.460	0.0739	0.611
1	—	—	—	—	0.0815	0.611

The experimental results are shown in figures (3) and (4). All the experiments were carried out with the temperature of the heating surface higher than the room temperature by about 94 to 134 deg F.

In the velocity distribution traverses no symmetry of flow between the plates was noticed. An example is shown in figure 5, where the gap was $\frac{1}{4}$ inch, and the discharge 590 pounds per hour.

Average velocities at radii $4\frac{5}{8}$ and $7\frac{7}{8}$ inches were calculated from the velocity distribution curves and from the quantity of air discharge as given by the calibration curve of the nozzle. Agreement was found within +2.7 and -2.3 percent.

ANALYSIS AND DISCUSSION OF RESULTS

Critical Reynolds Number :

It is noticed that the critical Reynolds number at which the laminar flow ceases to continue is not constant. The variation of that critical value with the ratio $\left(\frac{b}{L}\right)$ is shown by the solid line in Figure 6.

On the same graph the dashed line shows the critical R_e required to start turbulence.

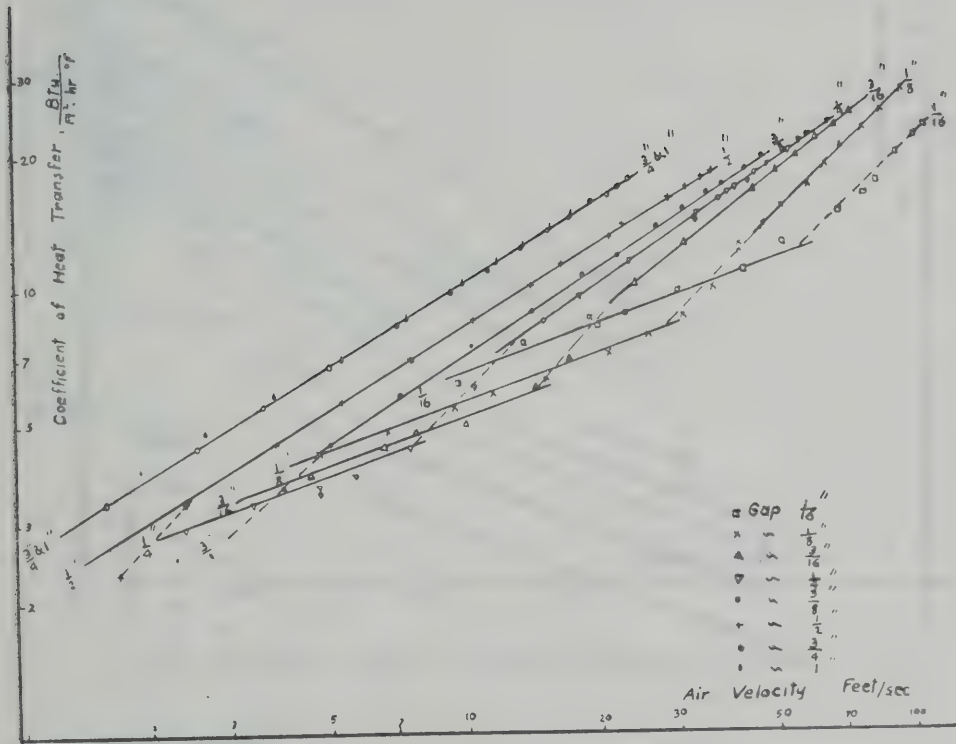


FIG. 3.—Heat Transfer Results for Different Gaps.

Similar results were obtained by Davies and White⁽²⁾ (*) who measured the friction when water was passed in very flat ducts. The ideas described by them and by Goldstein⁽⁴⁾ lead to the conclusion that the early transition with wider gaps is due to the fact that the parabolic flow is not established yet. As the gap increases more eddying is likely to occur. The conditions approach those of flow past flat plates. When the gap decreases, however, the parabolic flow has more chance to form. Turbulence is somewhat suppressed, and the critical value, therefore, increases.

(*) Numbers in parentheses refer to the references at the end of the paper.

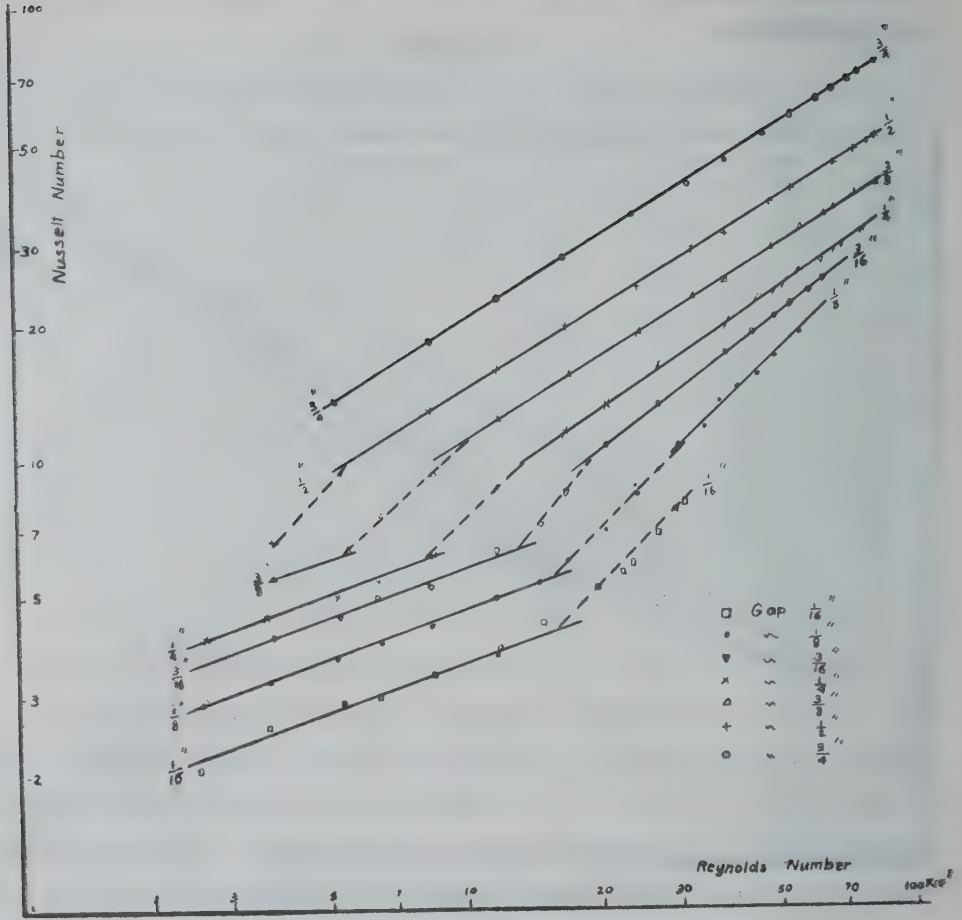


FIG. 4.—Heat Transfer Results for Different Gaps.

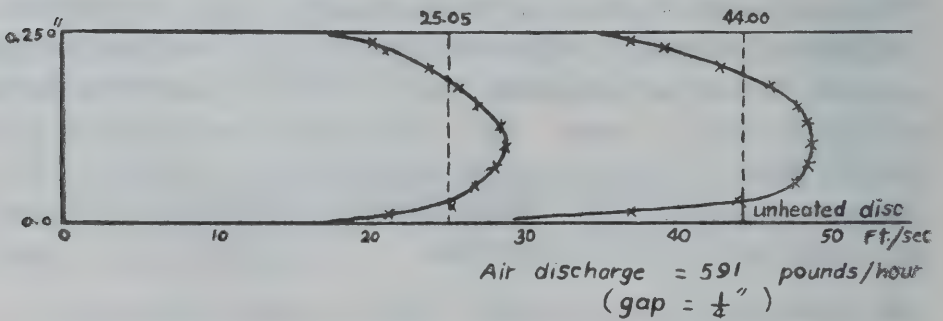


FIG. 5.—Velocity Distribution between the two Plates.

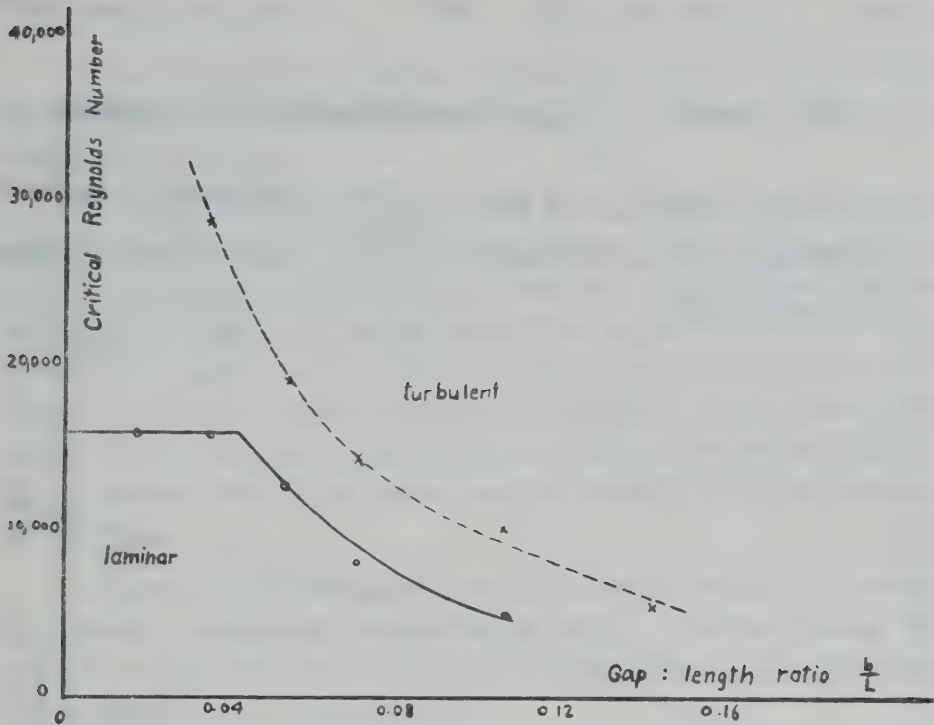


FIG 6.—Effect of Gap on Critical Reynolds Number.

Laminar Flow Results :

While the exponent (n) in the general equations (5) and (6) remains constant as the gap changes, the other constants K and A vary according to the laws

$$K = 1.233 \left(\frac{b}{L} \right)^{-0.2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\text{and} \quad A = 0.755 \left(\frac{b}{L} \right)^{0.44} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

After these two equations the general equations (5) and (6) for laminar flow become

$$h = 1.233 \left(\frac{b}{L} \right)^{-0.2} \cdot U_m^{0.36} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and
$$N_u = 0.755 \left(\frac{b}{L} \right)^{0.44} \cdot R_e^{0.36} \quad (10)$$

The values of $\left[\frac{N_u}{\left(\frac{b}{L} \right)^{0.44}} \right]$ were calculated from equation (10) and plotted versus R_e in Figure 7. The experimental values of N_u for different gaps were divided by $\left(\frac{b}{L} \right)^{0.44}$, and the results shown on the same graph.

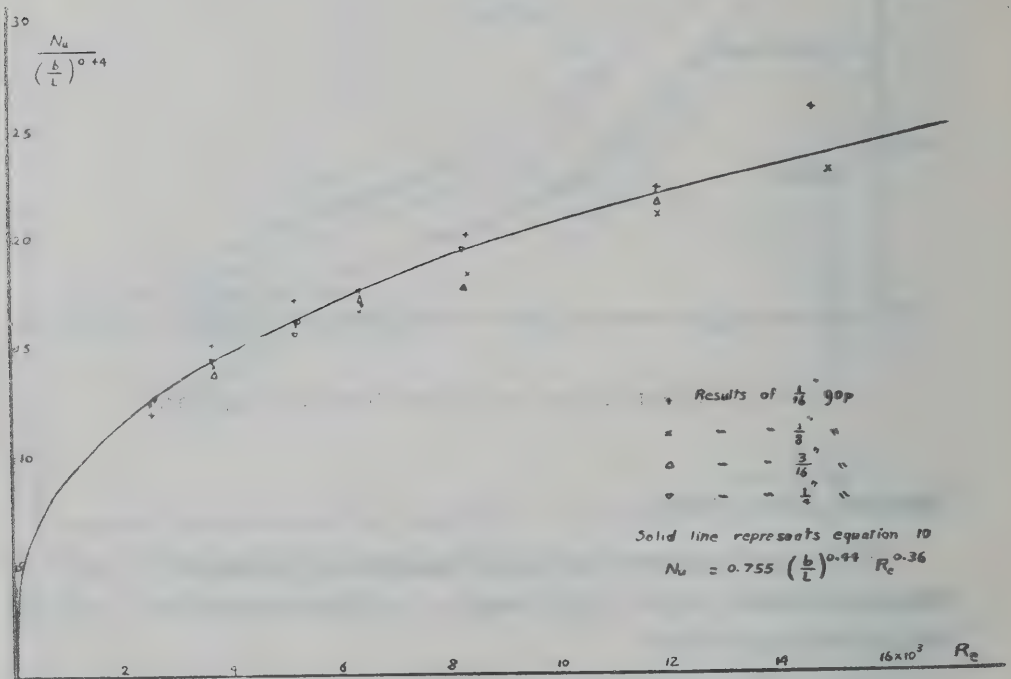


FIG. 7.—Correlation of Heat Transfer Results in Laminar Flow.

From equation (9) it is seen that the rate of heat transfer increases as the velocity increases and the gap decreases. This is quite natural because, in laminar flow of fluids, heat is transferred mainly by conduction. According to the equation of heat conduction the rate of heat transfer decreases with the increase of the distance travelled. The distance, in the present work, is the gap between the two plates.

Turbulent Flow Results :

In the turbulent flow range, the present results show, generally, an increase in the heat transfer as the gap (b) increases. This can be explained as follows :

Where a fluid enters a duct under certain conditions, eddies are set up. As the fluid proceeds along the duct, these eddies subside, and the stream assumes undisturbed flow with uniform heat transfer coefficient. These inlet eddies persist for longer distances in large ducts than in small ones, so that, for ducts of equal length (as in the present case), the mean heat transfer coefficient would tend to be higher for larger gaps. This effect tends to counteract the effect of the gap alone, which is to reduce the heat transfer with the increase in the gap.

In narrow ducts the proximity of the walls tends to minimize the velocity components normal to the walls. In other words, the freedom of the particles to move will be restricted, and turbulence suffers some kind of suppression.

Yet, this factor of more freedom in wide gaps is not unlimited. There may be a certain gap at which the effect of the depth of the air stream vanishes completely. As found in the present work this critical gap was $\frac{3}{4}$ inch. This corresponds to an $\frac{L}{b}$ ratio of 4.67.

The turbulent flow results for the different gaps in the present work can be represented by the equation

$$N_u = 0.075 R_e^{0.611} \cdot f \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$\text{where } f = 1 - 1.516 \left[1 - 4.67 \frac{b}{L} \right]^{0.8} \cdot R_e^{-0.06} \quad . \quad . \quad . \quad (12)$$

Figure 8 shows values of $\frac{N_u}{f}$ plotted against R_e according to equation (11) and (12). The results of the experiments on the six gaps ($\frac{1}{8}$ to $\frac{3}{4}$ inch) were corrected by dividing N_u by the value of (f)

for different gaps and Reynolds numbers. These corrected values are shown in Figure 8 so as to be compared with equations (11) and (12).

It is noticed also in the present turbulent flow results that the exponent of the $(N_u - R_e)$ curves varies from one gap to another, being greater as the gap decreases. Similar results were obtained by Lévêque⁽⁵⁾ and Washington and Marks⁽⁶⁾. El-sayed⁽³⁾ also found that the effect of reducing the length in the direction of the flow and the gap between solid boundaries on the turbulent velocity profiles, is to give the profiles transitional shapes between the laminar and turbulent distributions.

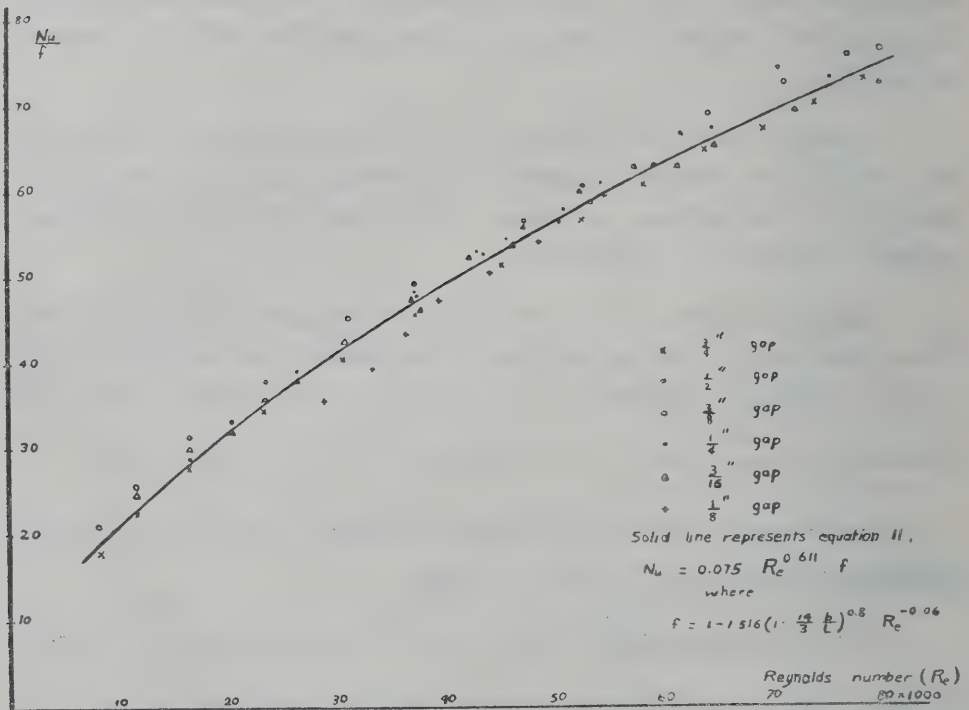


FIG. 8.—Results of Experiments for Turbulent flow Compared with Equ. 11

The length of the present heating surface (difference between radii) is small. In the inlet length the boundary layer at each wall is still growing, and the central part is being accelerated. As shown by Deissler⁽¹⁾ and confirmed by the present velocity distribution measurements, the growth of the boundary layer is slower for larger

Reynolds numbers. This means that, for the same gap, the inlet length increases as R_e increases. When the gap between the plates is sufficiently large all the heating surface lies in the entrance region. Therefore, the ratio of the inlet length to total length ($= 1$) will not be affected by the increase in R_e . The increase in N_u with R_e will be slower than for the fully developed turbulent part. This explains the constant exponent in the largest three gaps. However, when the gap decreases the ratio of the inlet length to the total length becomes smaller than one. This ratio increases as R_e increases, and, accordingly, the increase in N_u will be faster than for large gaps.

To sum up it may be stated that when the gap is small an increase in R_e causes an increase in the inlet length. This gives rise to heat transfer at a higher rate than with large gaps, as, in the latter case, no increase in the inlet length results from increasing the Reynolds number.

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It should be noted that, because of the limitations of space, some details have been omitted in this paper. The writer will be pleased to supply any further information upon request.

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GAS-TURBINE PERFORMANCE UNDER VARYING AMBIENT CONDITIONS

- I.—The Effect of the variation in Ambient Temperature on the power consumed by the compressor of a gas-turbine unit.
- II.—The use of Underground water in Precoolers for Industrial Gas-Turbines working under Tropical conditions.

BY

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I.—THE EFFECT OF THE VARIATION IN AMBIENT TEMPERATURE ON THE POWER CONSUMED BY THE COMPRESSOR OF A GAS-TURBINE UNIT

It is a well-known fact that the gas-turbine has already passed its critical development period and by passing the test of the operator acceptance has established itself as the most suitable power-generating machine in more than one industrial application. Many new applications, including atomic energy, make the future look extremely bright for the industrial Gas-Turbine.

Looking at the past, however, one can see that the gas-turbine did not become a success before many problems, both expected and unexpected, which crossed the path of its development, have been solved. We must not, however, feel too complacent. It is true that the gas-turbine is here to stay, but with many new applications in the process of being placed in service and others contemplated, new problems are bound to crop up which require a solution in order to have a reliable machine.

In its basic form, the internal-combustion gas-turbine consists of an air compressor, one or more combustion chambers, and a turbine.

The compressed air is heated by burning fuel in it or by exchanging heat with hot gases and expanded through the turbine which drives the air compressor and the load. In applications where a certain particular load requires a variable speed prime-mover or in others where a variable load requires a constant speed one, the turbine is usually divided into two parts, one driving the air compressor, and the other driving the load.

Additional equipment can be used in the cycle if a high efficiency and/or high specific output is required. For instance, a heat exchanger placed in the turbine exhaust recovers heat and transfers it to the air entering the combustion system, thus reducing the amount of fuel burned. Sometimes, the compressor is divided into two components with an intercooler between them to reduce the work of the second compressor. Introducing a reheat combustion chamber between the turbine stages achieves a higher specific output. The addition of one or more such equipments to the basic cycle have to be done, however, after a careful and complete investigation of all the factors and considerations involved have been made, since, under certain circumstances such an addition might defeat its own purpose by lowering the efficiency or at least raising the cost of the gas turbine unit unproportionately.

Like a chain whose strength is as big as that of its weakest link, the gas-turbine unit depends for its efficiency, cost and performance on how efficient and reliable is each of its components.

With the gas-turbine having established itself as the newest prime-mover in the heavy-duty Industrial field, we believe that it would be an ideal power-generating machine for many of the new applications presently in the process of being placed in service in our country. This is because many of the requirements of such applications are admirably served by the gas-turbine. Among these are the following :

1. Ability to use low-cost fuel with reasonable economy. Incidentally this offsets the lower efficiency of the gas-turbine as compared with standard Diesel Engines, and enables it to compete favourably with the latter in many applications such as locomotive-drive etc.

2. The small size and weight of the gas-turbine which go some way to override its higher fuel consumption relative to other prime-movers.

3. No water or other provision for cooling is essentially required.
4. The flexibility of the gas-turbine *i.e.* the possibility of choosing the particular cycle and arrangement of components best adapted to the required duty.

One of the most important applications of Industrial gas-turbines is in the Electric utility Industry. In this case they are either used in connection with existing steam stations or as independent units. A number of them could be located in stations at the end of long transmission lines and in this case they will be ready to supply the electric load of the surrounding area in case of a breakdown in the transmission line. Moreover, when the transmission lines are operating near maximum capacity and could not meet new peak loads, the gas-turbines can be used to supply the extra load thus saving the cost of new lines. In existing load centres, gas turbine units could be used to supply energy during periods of maximum demand. The gas turbine is ideally suited for this type of service because it can be started quickly and placed on the line without any warm-up period. In conjunction with hydroelectric units, the gas-turbine can be used to supplement them during low-water periods. Again, this application is ideal because the standby charges for gas-turbines are low, and units can be started quickly and simply after being out of service for long periods. The addition of a gas-turbine to an existing steam station is also of particular interest. The exhaust gas from the gas-turbine generator heats the feed water for the existing station. In this way, with the same original equipment, the plant capability could be increased while the station heat rate is reduced.

In considering the use of gas-turbines in any of the applications to be placed in service in our country, however, the problem of the effect of the variation in ambient conditions on the performance of the gas-turbine unit will acquire special importance. In order to make full use of the many points in favour of the gas-turbine under all prevalent ambient conditions, such an effect should be carefully and fully investigated.

It is for this purpose that the present investigation was carried out. The results recorded here should enable both the designer and operator to predict the behaviour of the gas-turbine unit under varying ambient conditions.

The component of a gas-turbine unit which would be directly affected by the variation in ambient conditions is the compressor. The net effect on the unit would naturally depend on the cycle used and the arrangement of the components. Once the effect on the compressor is known, the net effect on the unit could then be determined by simply following the arrangement of the components and evaluating the cumulative effect resulting from the change in the compressor's performance.

The present investigation was therefore limited to the effect on the compressor's performance. In order, however, to ensure that the results could be of universal use, the investigation was made to cover all the practical values of pressure ratios and efficiencies of a gas-turbine compressor and for all probable values of ambient conditions. These cover all the parameters on which the power absorbed by the compressor depend.

If, R is the pressure ratio of compression.

T_o „ „ absolute temperature of air before compression.

T_c „ „ „ „ „ „ at the end of compression.

η_c „ „ Isentropic Efficiency of compression.

γ „ „ Ratio of specific heats C_p/C_v .

$C_{p \text{ mean}}$ is the mean specific heat between T_o and T_c defined

$$\text{as } C_{p \text{ mean}} = (h_{T_c} - h_{T_o}) / (T_c - T_o)$$

where h_T is the Enthalpy at the temperature T .

then the work of compression per unit weight of air would be given by

$$\begin{aligned} W_c &= C_{p \text{ mean}} (T_c - T_o) / \eta_c \\ &= (h_{T_c} - h_{T_o}) / \eta_c \end{aligned}$$

The final compression temperature $T_c = T_o(R^{\gamma-1/\gamma} - 1)/\eta_c + T_o$.

Using the "Gas Tables" for Air at low pressures h_{T_o} was obtained for the ambient temperature T_o and then h_{T_c} was readily obtained for the value of compression ratio R under consideration. $(h_{T_c} - h_{T_o})$ will then give the Isentropic work of compression taking into account the variation of specific heat with temperature.

The actual work consumed in compression will depend as shown in the equations on the Isentropic Efficiency of compression η_c . For actual compressors, nowadays, the value of η_c ranges from 84 % to 86 %. W_c was tabulated for three values of η_c , namely, 84 %, 85 %, and 86 %.

The final temperature of compression T_c was determined and recorded so that it will be readily available for the further calculations on the actual cycle under consideration.

The results of the investigation are tabulated in Tables I, II and III, for the values of η_c of 84 %, 85 % and 86 % respectively.

The values of the ambient temperature T_o considered are shown in the Tables and they cover the range of all possible values, the lower values for temperate regions and the higher values for tropical regions.

The values of R considered are supposed to cover all practical values of pressure ratios for a single stage of compression.

Both linear interpolations and extrapolations from tables are possible when necessary giving values subject to insignificant errors.

The authors believe that data for Gas-Turbine Compressors working under any practical specified conditions in all probable regions will lie within the scope of these tables. The prediction of the performance of a gas-turbine unit under any specified conditions would be a simple matter.

TABLE I
Compressor Isentropic Efficiency = 84 %

R	T°R O°F	480	492	500	510	520	530	540	550	560	570	580	590
		20.3	32.3	40.3	50.3	60.3	70.3	80.3	90.3	100.3	110.3	120.3	130.3
2.0	W _c	30.3	30.9	31.4	31.9	32.5	33.2	33.8	34.4	35.0	35.6	36.2	36.8
	T _c	145.1	160.8	170.8	182.6	195.4	208.4	220.8	233.2	245.6	258.0	270.7	282.7
2.5	W _c	41.15	42.1	42.8	43.7	44.4	45.2	46.1	47.0	47.9	48.7	49.5	50.3
	T _c	191.6	207.4	218.0	231.9	244.7	258.0	271.6	284.8	298.8	311.7	325.0	338.3
3.0	W _c	50.6	51.8°	52.7	53.8	54.7	55.8	57.0	58.0	59.0	60.0	61.2	62.0
	T _c	230.7	247.5	259.1	273.6	287.1	301.6	316.5	320.7	344.5	358.5	373.0	386.3
3.5	W _c	59.0	60.4	61.3	62.7	63.8	65.2	66.3	67.5	68.8	70.0	71.2	72.5
	T _c	265.4	283.0	294.5	310.7	324.5	340.4	354.5	369.4	384.7	399.3	414.0	429.1
4.0	W _c	66.7	68.3	69.5	70.7	72.2	73.5	74.9	76.3	77.7	79.0	80.3	81.7
	T _c	297.1	315.6	328.5	343.2	359.1	374.4	389.7	405.3	421.0	436.0	450.6	466.4
4.5	W _c	73.6	75.4	76.7	78.2	79.8	81.2	82.7	84.2	85.7	87.2	88.8	90.2
	T _c	325.6	344.8	357.8	373.9	390.3	405.9	421.7	437.6	453.5	469.3	485.1	500.8
5.0	W _c	80.0	82.0	83.3	85.0	86.7	88.3	89.8	91.6	93.3	94.8	96.4	98.1
	T _c	351.8	371.8	385.0	401.7	418.5	434.8	450.6	467.6	484.2	500.0	516.0	532.7
5.5	W _c	86.8	88.2	89.7	91.3	93.1	94.7	96.5	98.3	100.1	101.9	103.7	105.6
	T _c	376.4	397.1	410.9	427.4	444.5	460.8	477.8	494.1	511.7	528.7	545.5	562.8

6.0	W_a	91.6	93.9	95.2	97.3	99.2	101.0	102.8	104.8	106.8	108.3	110.5	112.0
	T_a	399.3	420.1	433.5	451.8	469.5	486.3	503.2	520.9	538.5	554.7	572.9	588.3
6.5	W_c	96.8	99.3	101.2	102.8	104.8	106.7	108.9	110.5	112.6	114.5	116.7	118.7
	T_c	420.5	442.1	458.1	474.0	491.9	509.3	527.8	544.5	562.7	579.2	597.5	615.1
7.0	W_e	101.8	104.3	106.1	108.0	110.0	112.0	114.2	116.3	118.5	120.7	122.6	124.8
	T_c	440.6	462.7	477.8	496.1	512.9	530.6	549.1	567.1	585.5	604.0	621.0	639.3
7.5	W_o	106.8	109.2	111.0	113.0	115.3	117.3	119.5	121.8	124.0	126.5	128.5	130.7
	T_c	461.0	482.6	497.6	515.3	534.2	552.0	570.3	589.1	607.5	627.1	649.3	662.8
8.0	W_a	111.2	113.7	115.3	117.8	120.0	122.3	124.8	127.0	129.1	131.0	134.0	136.1
	T_e	479.0	500.7	514.9	534.7	553.1	572.0	591.5	609.9	627.9	646.9	666.3	684.2

W_o B.T.U. / lb.

T_e °F

TABLE II
Compressor Isentropic Efficiency = 85 %

R	T°R O°F	480 20.3	492 32.3	500 40.3	510 50.3	520 60.3	530 70.3	540 80.3	550 90.3	560 100.3	570 110.3	580 120.3	590 130.3
2.0	W _c	29.7	30.5	31.0	31.6	32.1	32.9	33.5	34.0	34.6	35.2	35.7	36.4
	T _c	144.0	159.3	169.1	181.8	193.9	206.9	219.3	231.5	243.8	256.3	268.3	281.1
2.5	W _c	40.7	41.6	42.3	43.2	43.9	44.7	45.6	46.5	47.5	48.2	48.8	49.7
	T _c	189.5	205.4	216.0	229.9	242.7	255.9	269.5	282.8	296.6	309.9	322.3	335.9
3.0	W _c	30.1	51.3	52.1	53.1	54.1	55.2	56.2	57.3	58.3	59.3	60.4	61.3
	T _c	228.7	245.4	256.7	270.6	284.8	299.2	313.2	327.4	341.4	355.5	369.8	383.4
3.5	W _c	58.3	59.8	60.7	62.0	63.0	64.3	65.5	66.7	68.0	69.3	70.4	71.6
	T _c	262.6	280.4	292.1	307.5	321.4	326.7	351.4	366.1	381.5	396.4	411.1	425.8
4.0	W _c	65.9	67.4	68.7	69.9	71.3	72.7	74.0	75.3	76.7	78.1	79.3	80.7
	T _c	293.8	311.9	325.0	340.0	355.5	371.1	386.2	401.3	416.8	432.3	447.0	462.4
4.5	W _c	72.8	74.6	75.8	77.3	78.8	80.2	81.7	83.2	84.7	86.2	87.7	89.2
	T _c	322.3	341.5	354.0	370.3	386.2	401.8	417.7	433.5	449.1	465.2	481.0	496.8
5.0	W _c	79.2	81.1	82.3	84.0	85.6	87.2	88.8	90.5	92.2	93.7	95.3	97.0
	T _c	343.6	368.1	381.0	397.6	414.2	430.3	446.6	463.0	479.8	495.5	511.0	528.3
5.5	W _c	85.0	87.2	88.6	90.2	92.0	93.6	95.4	97.2	98.9	100.7	102.3	104.2
	T _c	372.3	393.1	406.4	422.9	440.1	456.3	473.3	490.3	506.9	523.8	539.9	557.2

6.0	W_c	90.6	92.0	94.1	96.2	98.1	99.8	101.7	103.3	105.3	107.1	109.1	110.7
	T_c	395.1	416.0	429.1	447.4	464.8	481.9	498.7	514.7	522.6	549.5	567.1	583.2
6.5	W_c	95.8	98.2	100.1	101.7	103.5	105.5	107.7	109.3	111.5	113.2	115.3	117.2
	T_c	416.5	437.8	453.6	469.7	486.7	504.5	522.9	539.1	557.5	574.0	591.9	609.2
7.0	W_c	100.7	103.2	104.8	106.7	108.7	110.8	112.8	115.0	117.0	119.2	121.2	123.2
	T_c	436.4	458.2	472.5	489.9	507.6	525.8	543.6	562.0	579.6	598.0	615.5	633.1
7.5	W_c	105.3	108.0	109.8	111.7	113.0	116.0	118.5	120.4	122.8	125.0	126.9	128.9
	T_c	455.1	477.7	492.6	510.1	528.1	546.7	566.4	583.5	602.8	621.1	638.2	655.7
8.0	W_c	110.0	112.3	114.1	116.3	118.7	121.0	123.3	125.6	127.7	130.0	132.2	134.4
	T_c	474.1	495.0	510.0	528.6	547.9	566.8	585.5	604.3	622.3	640.9	659.6	677.4

W_c B. T. U. / lb

T_c °F

TABLE III
Compressor Isentropic Efficiency 86 %

R	T°R O°F	480	492	500	510	520	530	540	550	560	570	580	590
		20.3	32.3	40.2	50.3	60.3	70.3	80.3	90.3	100.3	110.3	120.3	130.3
2.0	W _c	29.4	30.2	30.6	31.2	31.8	32.5	33.1	33.6	34.2	34.8	35.4	36.0
	T _c	142.5	157.8	167.6	179.9	192.4	205.3	217.6	229.9	241.8	254.7	266.8	279.4
2.5	W _c	40.2	41.1	41.8	42.8	43.4	44.2	45.2	45.9	46.8	47.6	48.3	49.2
	T _c	187.6	203.3	214.0	227.8	240.7	253.9	267.8	280.6	293.8	307.5	319.8	333.8
3.0	W _c	49.5	50.7	51.5	52.5	53.5	54.5	55.6	56.3	57.5	58.6	59.6	60.5
	T _c	226.0	242.8	254.3	267.7	281.8	296.3	310.7	324.3	338.3	352.5	366.4	380.2
3.5	W _c	57.7	59.1	60.0	61.2	62.3	63.5	64.7	65.9	67.2	68.3	69.6	70.7
	T _c	260.0	277.7	289.3	303.9	318.7	333.4	348.1	363.0	378.0	392.7	407.4	422.2
4.0	W _c	65.2	66.7	67.8	69.1	70.5	71.8	73.2	74.4	75.8	77.2	78.4	79.8
	T _c	290.8	308.8	321.5	336.7	352.2	367.4	383.1	397.7	412.7	428.7	443.3	458.8
4.5	W _c	72.0	73.7	74.8	76.3	77.9	79.3	80.8	82.2	83.6	85.2	86.7	88.1
	T _c	319.0	337.8	350.1	366.1	382.5	398.1	414.0	429.4	444.9	461.1	476.9	492.3
5.0	W _c	78.3	80.2	81.3	83.0	84.7	86.2	87.7	89.4	91.1	92.7	94.2	95.8
	T _c	345.9	365.0	376.9	393.5	410.3	426.3	442.0	458.8	472.5	491.5	507.2	523.4
5.5	W _c	84.0	86.2	87.6	89.2	91.0	92.5	94.2	96.1	97.8	99.5	101.2	103.0
	T _c	368.3	389.1	402.4	418.8	435.9	451.9	468.4	485.8	502.4	518.9	535.4	552.3

6.0	W _c	89.6	91.8	93.0	95.0	96.8	98.8	100.4	102.2	104.1	106.0	107.8	109.6
	T _c	391.1	411.8	424.6	442.5	459.6	477.2	403.5	510.6	527.8	544.9	561.9	578.8
6.5	W _c	94.7	97.0	98.9	100.3	102.3	104.3	106.3	108.2	110.3	112.0	113.8	116.0
	T _c	411.9	433.1	448.7	465.0	481.8	499.6	517.4	534.7	552.8	569.1	585.9	604.3
7.0	W _c	99.6	102.0	103.7	105.3	107.5	109.7	111.5	113.8	115.8	117.0	119.7	121.8
	T _c	431.5	453.3	468.1	484.3	502.8	521.4	538.3	557.1	574.8	592.8	609.6	627.5
7.5	W _c	104.2	106.8	108.2	110.3	112.7	114.7	117.8	119.0	121.3	123.5	125.3	127.5
	T _c	450.4	472.6	486.3	504.4	523.8	541.5	560.4	578.0	596.8	615.1	631.8	650.1
8.0	W _c	108.8	116.0	112.8	115.1	117.2	119.5	121.8	124.0	126.1	128.3	130.8	133.0
	T _c	469.3	490.1	514.9	523.8	541.9	560.8	579.6	597.9	615.9	634.3	653.7	671.9

W_c B.T.U./lb.

T_c °F

II.—THE USE OF UNDERGROUND WATER IN PRECOOLERS FOR INDUSTRIAL GAS-TURBINES WORKING UNDER TROPICAL CONDITIONS

In Part I of this investigation, it was made clear by the tabulated figures that the effect on the power consumed by a gas-turbine compressor of the variation in ambient temperature is considerable. The same general result was shown before in the study of particular cycles working under certain specified conditions, although a complete and comprehensive analysis of the problem as was done here does not appear in the literature.

In order to demonstrate the net effect on the performance of a gas-turbine unit of the variation in ambient temperature, the authors have chosen here a cycle which they think will find in the future a wide scope of application not only in this country but also in most tropical countries having ample resources of fuel oil. This is the cycle most suited for Peak Loads. The arrangement for such a cycle is shown in Fig. 1. It is a simple cycle with a Heat Exchanger. It is ideal for this application, namely Peak Loads, for the following reasons :

1. Relatively low capital cost since it comprises the least number of components consistent with a reasonable overall efficiency.
2. It can be started very quickly with the least number of personnel and least supervision.
3. Space and maintenance are minimum.

Calculations were made for the following cycle data :

1. Inlet temperature to turbine $T_{\text{max.}} = 800^{\circ}\text{C}$ (1472°F).
2. Compressor pressure ratio = 6.
3. Compressor Isentropic Efficiency = 85 %.
4. Turbine Isentropic Efficiency = 86 %.
5. Heat Exchanger Thermal Ratio = 75 %.
6. Mechanical Efficiency of Compressor = 99 %.
7. Mechanical Efficiency of Turbine = 99 %.

8. Pressure Losses :

- (a) Compressor intake loss = 1 % .
- (b) Heat Exchanger loss: (i) Air side 1.5 % .
(ii) Gas side 1.5 % .
- (c) Combustion Chamber loss = 2.5 % .
- (d) Exhaust loss = 1 % .

9. Combustion Chamber Efficiency = 98 % .

The range of Ambient Temperatures which will be of special importance in dealing with gas-turbines working under tropical conditions could generally be taken as from 50°F to 130°F (10°C to 55°C). Table IV shows how the Specific Output (Output per unit weight of air consumed) and Overall Thermal Efficiency of the cycle under consideration vary for the range 50°F to 130°F as well as for the lower temperature range of importance in temperate countries *i.e.* between 20°F and 50°F.

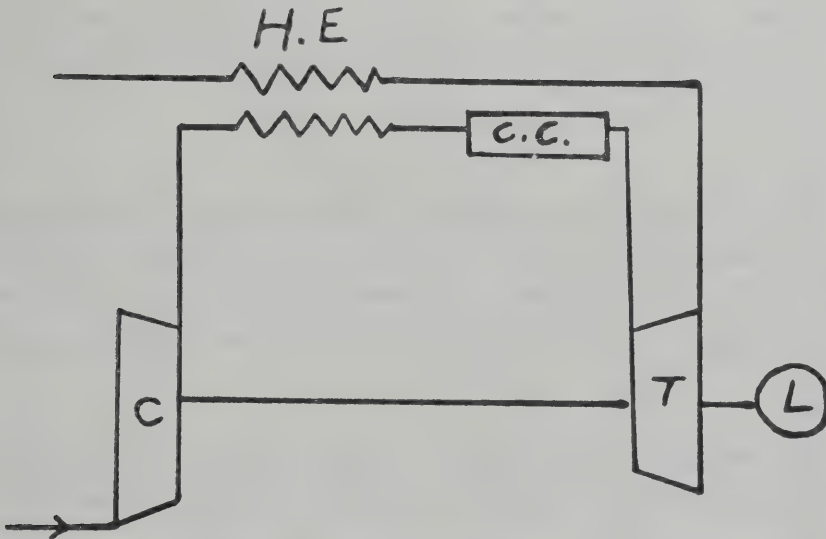


FIG. 1

C	Compressor	T	Turbine
CC	Combustion chamber	H E	Heat Exchanger
L	Load		

TABLE IV

T_o		W	η_o	W %	η_o %
$^{\circ}\text{R}$	$^{\circ}\text{F}$				
480	20.3	65.3	33.7	109.8	107.7
492	32.3	63.1	32.8	106.0	105.0
500	40.3	61.7	32.2	103.6	103.0
510	50.3	59.6	31.4	100.0	100.0
520	60.3	57.8	30.6	97.1	97.5
530	70.3	55.8	29.7	93.7	94.6
540	80.3	54.1	29.0	90.8	92.5
550	90.3	52.4	28.3	87.9	90.0
560	100.3	50.4	27.2	84.7	86.7
570	110.3	48.6	26.5	81.6	84.5
580	120.3	46.6	25.5	77.2	81.3
590	130.3	44.98	24.8	75.5	79.1

W B.T.U./lb.

η_o Overall Thermal Efficiency.

For the sake of comparison, a datum has been chosen at the ambient temperature of 50.3°F , thus showing the relative change in the specific output W and the overall efficiency η_o . The relative specific output W % and relative efficiency η_o % are also tabulated in Table IV.

The table shows how considerable is the effect of the rise in ambient temperature on the performance of the cycle considered.

As an example, a gas-turbine unit operating on the simple regenerative cycle treated here and rated at an ambient temperature of say 50°F , would only give 81.6 % of its rated power output at an ambient temperature of 110°F and that at a decreased relative efficiency of 84.5 %.

Fig. 2 shows clearly the considerable drop in both the specific output and thermal efficiency with rising ambient temperature.

Now, if the power to be installed is reckoned at the lowest temperature, the unit will not give its rated output at a higher temperature. Conversely if the power is reckoned at the high temperature, the unit will then have to work at part load at the low temperature *i.e.* at a lower efficiency combined with unnecessary extra capital cost.

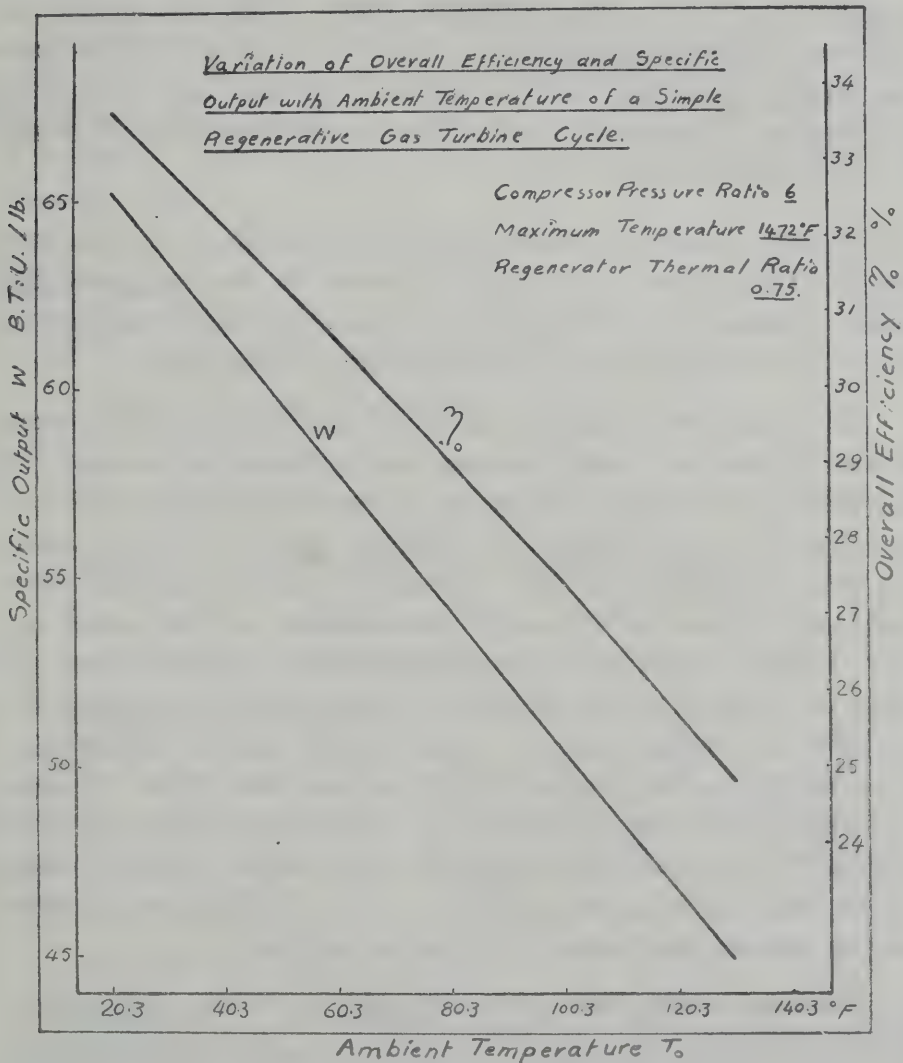


FIG. 2

The problem, then, would be to be able to keep the output of the unit as constant as possible irrespective of the variation in ambient temperature. The simplest solution which presents itself is by keeping the air inlet temperature to the compressor at its lowest value in a changing ambient temperature.

The methods suggested so far as a solution to this problem include the use of a refrigeration plant before the inlet to the compressor (driven by the turbine exhaust energy) or the injection of a volatile liquid into the compressor intake. The first method has not been exploited yet and it waits trial to find out whether the extra complication and cost will be justified by the improvement in performance. The second method has serious limitations in regard both to cost and scope.

A much simpler and much more economical method has presented itself to the authors. This simply consists of the use, under high ambient temperature conditions, of the naturally cooler underground water in precooling the air before entering the compressor.

In Egypt, and probably in most of the tropical regions, ample sources of this water exist at economically reasonable depths. The temperature of underground water remains invariably constant all over the year. In Egypt, for example, the underground water temperature is 24°C . (75.2°F). During the hot season, that is from April until September, the ambient temperature is much higher than 24°C , indeed it reaches an average of 40°C . A gas-turbine unit which is designed to give its full load at an ambient temperature of say 25°C . will suffer a drop in both specific output and efficiency when working under the prevailing ambient conditions. In order, therefore, to retain as completely as possible the rated output and efficiency of the unit, it is suggested as mentioned before to precool the air before entering the compressor. The precooler would have to use the underground water as the cooling medium.

The retainment of the power and efficiency of the unit would, of course, be done at the price of the extra cost of a precooler and the cost and power of the water pumping equipment to be installed.

The cost of a precooler, as known from practice, would be more than justified by the gain in both output and efficiency. The power of the water pumping equipment, and consequently its cost, would be as shown below, quite insignificant. There is, of course, the extra pressure loss in the precooler which will contribute to the lowering of the specific output of the unit. The following analysis, however, shows that the drop in specific output due to this cause is insignificant compared with the specific output regained by the use of the precooler.

Let us take, as an example, the gas-turbine unit treated before, namely that working on the simple regenerative cycle, and consider the addition to the cycle of a precooler in the way suggested in this investigation. For the sake of economy let the precooler be of the cross-flow type.

Assumptions :

Pressure loss in the precooler = 1.5 % .

Precooler Thermal Effectiveness = 90 % .

(If a contra-flow precooler was used the Effectiveness would be as high as 100 %).

Underground water temperature = 24°C. (75.2°F).

Total Manometric Head on Pump = 60 fts.

(considered nearly maximum value in Egypt).

Pump Efficiency = 75 % .

The power needed by the pump per lb. of air compressed, depends on the mass flow of water per lb. of air. According to the general practice of water-air heat exchangers, the mass flow of water per lb. of air is 1.21 lbs.

Power per lb. of air per sec. = 0.176 H.P.

Under ambient temperature of 110°F (43.5°C.), the specific output, *without using a precooler*, is equal to 48.64 B.T.U. / lb. (68.5 H.P. / lb. / sec.), at a thermal efficiency of 26.5 % .

The new specific output, *after using a precooler*, and after taking into consideration the extra pressure loss in it is equal to 53.25 B.T.U./lb. (75.3 H.P./lb./sec.). The net specific H.P. after deducting the pump power would be $75.3 - 0.176 = 75.124$ H.P./lb./sec., at an efficiency of 28.63 %.

The following table represents the regain in the specific output and overall efficiency by using the precooler.

WITHOUT PRECOOLER		WITH PRECOOLER
W B.T.U./lb.	48.64	53.2
H.P./lb./sec.	68.5	75.124
η_o %	26.5	28.63

It is quite obvious from the above analysis that a precooler operating in the very simple way suggested here represents a perfectly sound solution for the serious problem of the high ambient temperature in tropical countries which, no doubt, must have acted so far as a deterrent to the use of gas turbines in this part of the world. More work on these lines, both theoretical and practical, is recommended by the authors as they believe that the results more than justify the time and expense spent on it.

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THE COMBUSTION OF FUEL OIL DROPLETS

A Historical Review and a Plan for a Future Investigation

BY

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INTRODUCTION

The decline in world resources of fuel oils is a matter of great concern to both supplier and consumer. The most important source of power is being threatened by extinction. It is true that the advent of atomic power strikes a conciliatory note but it will certainly be a long time before this new source can replace completely the traditional ones. Until then, the problem of a pending shortage of fuel oils supply would have to be solved. Whatever are the suggested solutions, there is a general agreement on one point namely that the existing supplies should be used in the most economical way. Combustion efficiency should be brought up as high as possible. Of prime importance, therefore, to the engineer is an understanding of the process of burning fuels. The problem is of a particular importance in the Middle East for two reasons, *firstly*, there is the fact that future resources of fuel oils will be mainly Middle East crudes. *Secondly*, a big proportion of the yield of these crudes is composed mainly of residual type fuels whose process of burning is only slightly understood. Since the fuel oils are generally burnt in the form of sprays which consist of small droplets of varying size, it was natural that the greatest parts of study for a best understanding of their combustion have been made on single droplets.

In the present article, a review is first made of the work carried out so far on this point with the purpose of finding out the points which would need more study. An outline of a plan of study which is being carried out at present by the authors is then given.

HISTORICAL REVIEW

The combustion of sprays of liquid fuels involves a number of complex processes, most of which are only incompletely understood, but which could be summarised as follows:— As oil droplets enter the furnace, they receive heat by convection from preheated combustion air and from recycled combustion products. They also absorb energy radiated from the furnace walls and from the flame. This addition of heat causes the droplets to evaporate and in some cases to polymerise and crack while still being in the liquid state. The major portion of the combustion occurs in the gas phase with the production of a diffusion flame lying either at the oxygen-rich boundaries of eddies containing many vaporising droplets or, depending on mixing conditions, a diffusion flame at least partly surrounding each droplet. Also transition from the first to the second state may occur along the flame path in the furnace. This flame is often made luminous by the formation of carbon due to cracking reactions in the gas-phase.

It is clear that the rate of combustion of the fuel spray is normally considered a complicated function of the following:

1. The size and velocity distribution of the droplets.
2. The physical and chemical properties of the fuel.
3. The surroundings (gas composition, temperature, etc.).
4. The character of the mixing process of the spray with air and with combustion products.

Recent papers have reported certain experimental and theoretical aspects of combustion of liquid droplets in gaseous reactants. The experimental procedures employed in these fundamental studies of the burning rate may be grouped into the following three categories:

A.—The process of non stationary, nonsteady-state combustion in which a free falling droplet is allowed to come in contact with

a reacting gas (such as the works of Topps (5), Hottel and Simpson (8), Ezzat (10)).

B.—The process of stationary, nonsteady state combustion in which the combustion rate of a droplet suspended in a reacting medium is determined from the variation of droplet size with time (such as works of Godsave (2), Hall and Diederichen (1), Kiyosi Kobayasi (6), Niichi Nishiwaki (7)).

C.—The process of stationary, steady state combustion in which the geometric dimensions of a supported droplet are maintained constant during combustion (such as works of Splading (3), Wise and Wood (4)).

The Theory of Combustion of a Liquid Fuel Oil Droplet.

The Film Concept and Combustion of Liquids.

The combustion of liquids is an example of combined heat transfer, evaporation, diffusion and chemical reaction. Much work has been done on the related problem of heat and matter transfer in all of which a thin film of stagnant gas was visualised as adhering to the surface of the liquid across which heat transfer takes place by pure conduction.

The whole temperature drop was also considered to occur across this thin film. This film concept is illustrated in (figure 1) where the appropriate temperature and partial pressure curves are shown.

An accepted assumption by most of investigators is that there is no chemical resistance to combustion, which implies that as soon as fuel and oxygen molecules are brought together by diffusion they react producing the appropriate amount of heat. Although Kinetic considerations show that there must be several stages in the reaction and that numerous intermediate compounds and radicals are formed they are supposed to follow one another so quickly that they occupy negligible space and time. The stagnant film is therefore divided into two parts by a surface parallel to the film boundaries at which combustion takes place. This surface is represented by (B) whilst the film boundaries are represented by (A) on the liquid side and (C) on the gas side.

In the part of the film between (A) and (B) fuel diffuses as vapour from the liquid side, while oxygen diffuses from the gas stream beyond (C). These molecules are consumed at (B) and heat is developed while molecules of the combustion products are produced which must diffuse outwards across (B.C.). These combustion products are assumed in the first instance to be carbon dioxide and water vapour, this neglects dissociation which will certainly be considerable at the temperatures in question, though difficult to assess quantitatively.

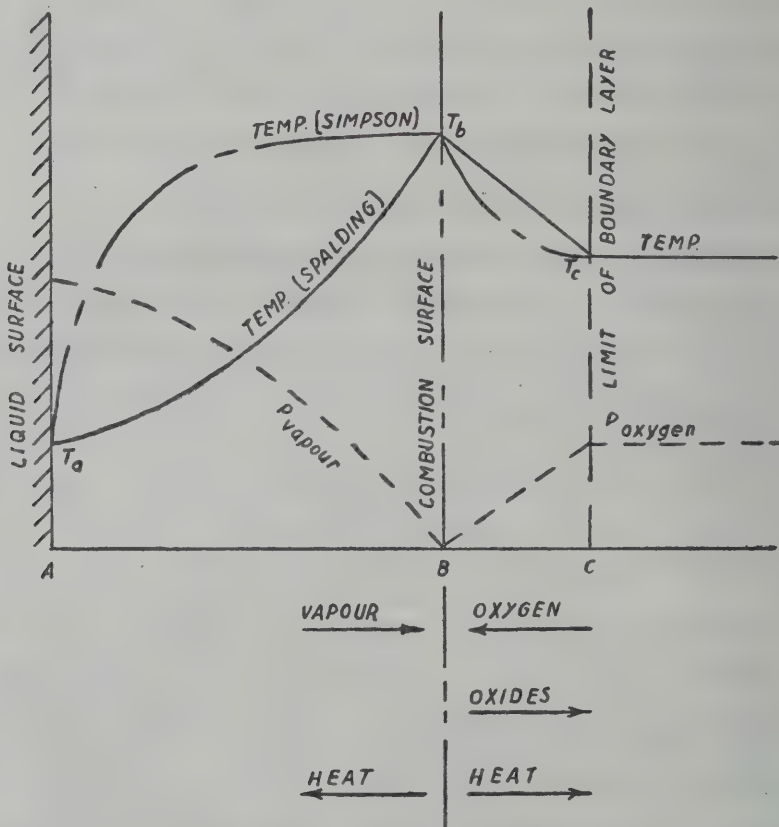


FIG. 1.—Temperature and Matter Distributions on Burning Fuel Oil Droplets in a Gas Stream

Under steady conditions, the heat developed at (B) must be dissipated as soon as produced. For this the only mechanism is conduction, whilst radiation as stated by most of investigators is considered negligible in comparison to conducted heat.

Combustion Time :

The combustion time of separate droplets has been always the main objective of recent works in this field. All these works were based on the assumption that all the fuel arriving at the combustion surface is consumed. Then under steady state condition the rate of consumption (or burning) of fuel is equal to the rate of vaporisation. This leads to the fact that the prediction of reaction time is a physical problem and not a chemical one.

The rate of vaporisation was calculated by previous investigators by setting up equations for a heat and matter balance for the diffusing system, the quantities involved being :

- (a) The diffusion of oxygen to the combustion surface.
- (b) The fuel/oxygen ratio.
- (c) The heat flow between the combustion and liquid surfaces and that between the combustion surface and the ambient gas.

These equations give burning time depending on the evaporation rate which is controlled by :

1. The heat transfer rate through the diffusion film to the liquid surface.
2. The geometry of the surface.
3. The nature of the fluid.
4. The velocity of the oxidiser stream flowing past the droplet.

The results of all investigators in this respect verified that the rate of evaporation from a spherical surface is proportional to the first power of the diameter which is called by Godsave (2) "The first power of the diameter law". This was deduced as follows :

The plot of experimental results of the (diameter)² against (time) was found to be linear.

$$\text{Thus } d^2 = d_0^2 - \lambda t \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where d_0 is the diameter at time $t = 0$; and λ is a constant characterising the rate of decrease in size of the drop, and is called the evaporation constant which is determined directly from the slope of plots (d^2) against (t).

By differentiating equation (1) $2d \frac{dd}{dt} = -\lambda$

$$\begin{aligned} \text{But rate of change of volume } \frac{d}{dt} \left[\frac{4}{3} \pi \left(\frac{d}{2} \right)^3 \right] &= \pi \frac{d^2}{2} \frac{dd}{dt} \\ &= -\frac{\lambda \pi}{4} d \end{aligned}$$

At constant mean density of the liquid (ρ) it follows that the mass rate of change in size is

$$\frac{dm}{dt} = -\rho \frac{\pi \lambda}{4} d \quad (2)$$

Again from equation (1), if (t_1) is the drop life in seconds it follows that $d = 0$ when $t = t_1$ hence $t_1 = \frac{d^2 \rho}{\lambda}$ which means that the droplet combustion time is proportional to the square of the initial diameter.

RECENT EXPERIMENTS

Recent experiments were carried out by one of the authors (Mr. M.I. Mikhail⁽¹⁾) on different types of hydrocarbons namely straight distillates, residual fuels as well as blends of both types using a technique similar to that used by Godsave (2), in his investigations referred to herein by suspending droplets on silica filament and then filming the droplets burning. Here, radiating elements were used in order to simulate by their radiation of heat the true condition under which droplets burn inside the body of industrial flames. Incidentally these radiating elements function also as igniter to droplet instead of igniting torches or high tension sparks used by other investigators.

(1) Professor M.W. Thring, professor of fuel Technology at Sheffield University has kindly helped in providing the facilities for carrying out these experiments.

The results of these experiments, reference figure (No.2) showed the following :

A.—*Distillate type fuels :*

- i. A linear relation between evaporation rate and time, which agrees with the “First power of the diameter law” originally developed by Godsave (2).
- ii. The recorded rate of evaporation for Kerosine which was taken as a reference fuel for comparison showed an increase of 48 % over that given by Godsave (2) in his experiments on same fuel.

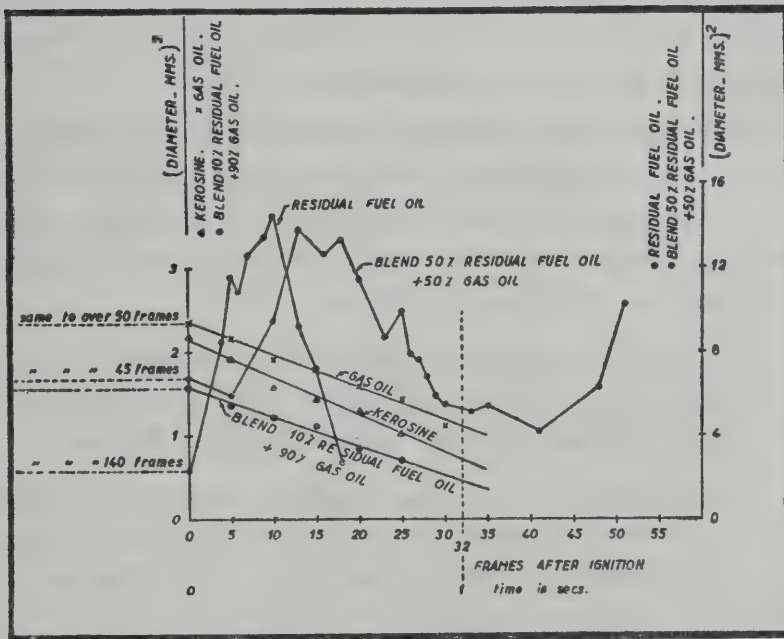


FIG. 2.—Burning Characteristics of Fuel Oil Droplets

B.—*Residual type fuels :*

- i. Burning of residual fuel droplets behaved differently. Droplets of these fuels swell during in contrast with straight distillate type fuels, and the burning in this case does not follow the same law i.e. “First power of the diameter law”. Professor

M.W. Thring has shown great interest in these recent experiments and commented on the results obtained stating that further work should be carried out on same lines to clarify points so far not understood.

FUTURE INVESTIGATIONS

After a study of previous investigations carried out on burning of fuel oil droplets, a survey of which is stated herein, the authors believe that the results fail to explain certain phases of the burning process which they think need further study before a complete and comprehensive understanding of the droplet burning is obtained. The points which will be therefore covered in the present phase of experiments could be briefly summarised as follows :

1. Radiation effect of droplet own flame on the burning time. In order to show this effect the authors apply in their current experiments a technique which they adopted and that simply consists of evaporating a droplet in a Nitrogen atmosphere thus eliminating the factor of radiation of droplet own flame.

2. Radiation effect of other surrounding droplets flames on the burning time of the investigated droplet. Here, radiating elements will be used to simulate the effect of other burning droplets.

3. The condition of droplet interior during the process of burning, and those conditions which lead to the formation of cenospheres on burning residual type fuels. The technique used by the authors in this case consists of hanging the droplet at the end of a tube carrier through which thermocouple wires are lead to the centre of the droplet.

4. The effect of the surrounding gas composition on the burning time. This will be achieved by burning the droplet in an atmosphere of gas of controlled composition.

5. The effect of the flame envelope thickness on the burning behaviour of the droplet.

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COMPOUNDING THE NORMALLY ASPIRATED SPARK IGNITION ENGINE

BY

Dr. TALAAT EL-SAYED AHMED YOUSSEF

The purpose of this paper is to focus the attention on the potentiality of power recovery of part of the energy in the exhaust of normally aspirated spark ignition engines. In view of the fact that from 30 to 50 % of the energy of the fuel released during combustion in an engine escapes through the exhaust, increasing attention is being paid to the recovery of this energy.

The energy in the exhaust can either be used for heating or for producing mechanical work. The thermal utilization of the exhaust has already been successfully applied to heat exchangers and exhaust boilers. The utilization of the energy in the exhaust in producing mechanical work is more tempting and rewarding since it adds directly to the output and thermal efficiency of the engine. Engines designed with power recovery of part of the exhaust energy are called compound engines.

Compounding has been given greater attention by aero engines engineers and as a matter of fact it is viewed by most of them as the last futile attempt to prolong the existence of the reciprocating engine for aircraft propulsion. Although jet stack arrangements⁽¹⁾ can be considered as a form of compounding the aero-engine, many compounding arrangements have been proposed lately⁽²⁾ covering a very wide range, all the way from the turbosupercharged aircraft

(1) See SAE Transactions Vol. 54, 1946 "Utilization of Exhaust Gas of Aircraft Engines", by B. Pinkel.

(2) See SAE Transactions Vol. 4, No. 1, 1950 "Compound Engine Systems for Aircraft", by E.J. Manganiello, L.V. Humble and D.S. Boman.

engine, where the engine develops all the shaft power, to the hot-gas generator plus turbine combination, where the turbine develops all the shaft power. These arrangements deal only with aircraft engines and the highly supercharged compression ignition engines⁽³⁾.

Computations presented in this paper will help to show the possibility of compounding the normally aspirated spark ignition engine. Only a blowdown turbine is to be connected to the engine crankshaft through proper gearing. The blowdown turbine is similar to the Buchi and Holzworth turbines. It derives power from the velocity energy generated during the exhaust blowdown period and requires individual stacks from each cylinder, or groups of cylinders without overlapping exhaust periods to the turbine nozzle box.

Because it utilizes the existing velocity energy of the exhaust gases, it does not impose any back pressure on the engine. A steady-flow turbine works on the expansion of the exhaust gases from the higher pressure in the exhaust pipe to the lower atmospheric pressure. The back pressure imposed by the steady-flow turbine, although can be utilized with advantage in case of highly supercharged engines or at high altitudes, yet it cannot be tolerated in the case of normally aspirated engines, especially under part load throttled conditions. It will be shown later that the blowdown turbine will contribute a considerable amount of useful work under partially opened throttle conditions.

CONVERTIBLE BLOWDOWN ENERGY

The reciprocating engine exhaust process is one of the most interesting and complicated problems encountered in engine analysis because it involves extremely rapid changes in thermodynamic and gasdynamic characteristics of the working fluid.

The theoretical Otto cycle indicator diagram is illustrated in Fig. 1 Process (4-1) is the release or blowdown process, where the exhaust gas is discharged from the cylinder. The maximum theoretical work

⁽³⁾ See SAE Transactions Vol. 3, No. 1949 "Compound Powerplants", by P. H. Schweitzer and J. K. Salisbury.

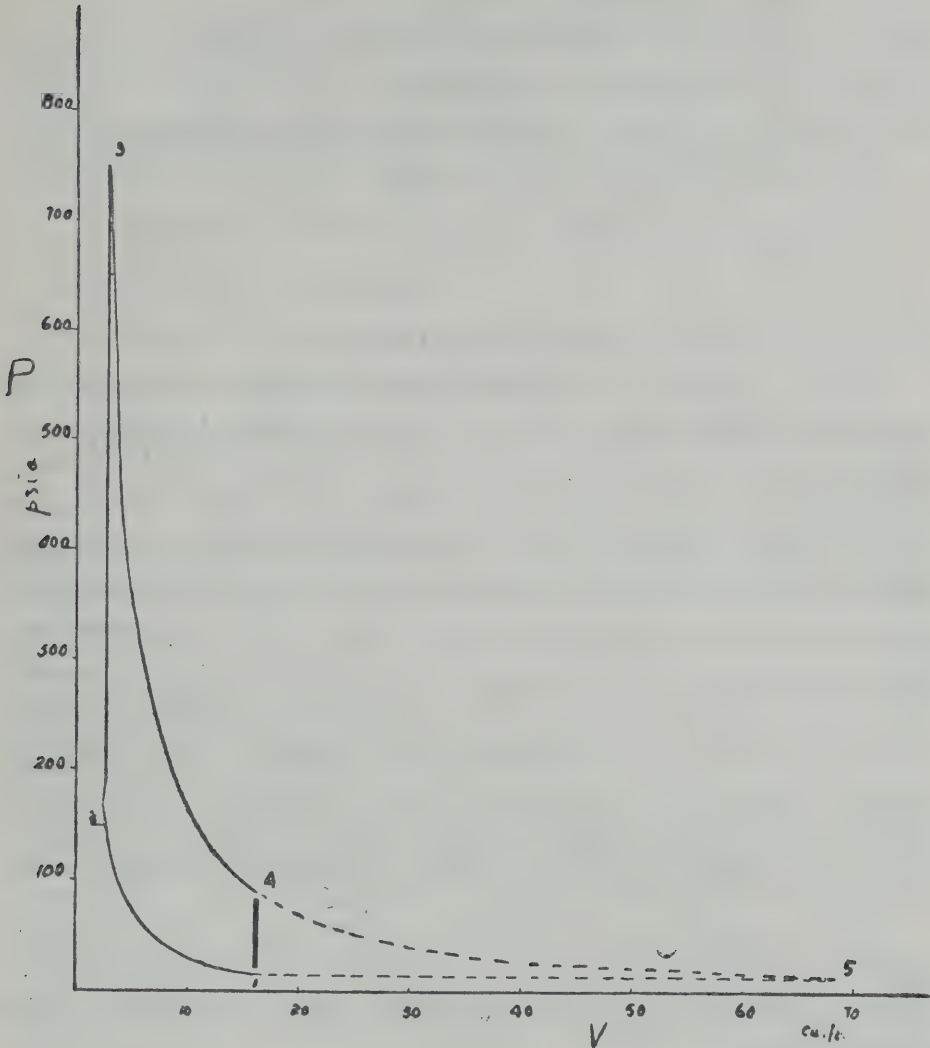


FIG. 1

that can be recovered from the exhaust gases in a blowdown turbine is represented by the tail of the P-V diagram. This maximum work that can theoretically be obtained from the exhaust gases during the blowdown period is called the convertible blowdown energy⁽⁴⁾ and is given by,

⁽⁴⁾ See "Scavenging of Two Stroke Cycle Diesel Engines", by P.H.Schweitzer.

$$W_e = (E_4 - E_5) - A p_1 (V_5 - V_1).$$

where W_e = Convertible blowdown energy Btu/lb.
 E = Internal energy Btu/lb.
 P_1 = Exhaust manifold pressure lb/sq. ft. absolute.
 V = Specific volume cuft/lb.
 $A = \frac{1}{J} = \frac{1}{778}.$

BLOWDOWN LOSSES

Only a portion of the convertible blowdown energy can be recovered as useful work because of the unavoidable blowdown losses. Radiation and conduction heat losses from the port and the exhaust pipe will reduce the energy content of the gases. The conversion of potential energy into directed kinetic energy is generally accompanied by friction, shock and eddy losses which reconvert part of the directed kinetic energy back into random kinetic energy (heat) at approximately constant pressure. At present, power recovery is not a consideration in the design of exhaust ports. Even if it were, the port design could be correct for only one instant during the blowdown, and would be either too large or too small for all other conditions. Consequently the reheat loss will be always present in the blowdown process, but it probably can be reduced somewhat by improved port design.

On the other hand the blowdown turbine has to negotiate widely varying blade to jet speed ratios, which will make the design of an efficient blowdown turbine rather difficult and involving considerable compromises.

PERFORMANCE CALCULATIONS

The influence of various design and operating parameters, as well as the performance of the blowdown compounded engine, can be predicted with fair accuracy by computations based on the thermodynamic charts ⁽⁵⁾.

⁽⁵⁾ See SAE Transactions Vol. 31, October 1936 "Thermodynamic Properties of Working Fluid in Internal Combustion Engines", by R.L. Hershy, J.E. Eberhardt and H.C. Hottel.

Design factors such as engine compression ration, induction manifold pressure and the efficiency of recovery of exhaust blowdown energy were studied and results presented herein are illustrative in showing the effect of each of these factors on engine performance. Assumptions used in the computations of theoretical cycles are:

1. Ambient air is at 14.7 psia and 100°F.
2. Processes are isentropic.
3. Mixture used is the chemically correct mixture of C_8H_{18} and air, fuel-air ratio .0665 (Hottel Charts).
4. Inlet and exhaust valves open and close at dead centres.

In view of the fact that test results on this type of compound engines are unavailable at present, the following assumptions are made in an attempt to arrive at the expected performance of actual engines.

1. 20 % of the heat added is rejected to the cooling water. The distribution of the cooling losses throughout the cycle is taken in the manner given by Taylor and Taylor ⁽⁶⁾.

2. Due to the fact that in an actual engine combustion does not take place at constant volume and the exhaust valve has to be opened considerably before dead centre, a diagram factor of .9 is taken.

3. A value of 81.5 % for the mechanical efficiency is taken as this represents the average of the mechanical efficiencies of a good number of modern automobile engines at maximum power R.P.M.

4. No attempt has been made to analyse blowdown losses. To study the gain in performance by the recovery of part of the energy in the exhaust, values varying from 70 to 30 % are assumed to represent the efficiency of blowdown recovery. These figures are supposed to cover the losses during the blowdown period as well as the inefficiency of the blowdown turbine.

(6) See "The Internal Combustion Engine", by C.F. Taylor. and E.S. Taylor

FULL THROTTLE PERFORMANCE

The percentage increase in net output and thermal efficiency of the theoretical cycle with blowdown compounding depends to a great extent on the compression ratio. The maximum convertible blowdown energy expressed as a fraction of the total energy in the fuel is shown in Fig. 3 to vary from .153 at a compression ratio of 4 to .112 at a compression ratio of 16. The convertible blowdown energy per pound of mixture decreases from 185 to 142 Btu while the piston work increases from 287 to 672 Btu through the previous range of compression ratio. The drop in convertible blowdown energy as the compression ratio is increased is due to the fact that longer expansion results in lower pressure and temperature at the moment of opening the exhaust valve.

Fig. 2 shows the percentage increase in power and efficiency with compounding. This increase decreases from a value of 62% at a compression ratio of 3 to 21.5 % at a compression ratio of 16 and indicates the very interesting fact that compounding is more rewarding at low compression ratios.

Fig. 2 shows also that at any compression ratio the IMEP of the theoretically compound cycle exceeds that of the Otto cycle by a value which varies from 70 psi at a compression ratio of 3 to 57 psi at a compression ratio of 16. The theoretical effect of compounding on efficiency and indicated fuel consumption is shown in Fig. 3. The theoretical cycle efficiency at a compression ratio of 4 increases from .305 to .458 with compounding. Such an efficiency can be reached only at a compression ratio of 10 with an Otto cycle. Similarly compounding an engine of a compression ratio of 6 will increase its efficiency to a value which will be attained by increasing its compression ratio to 16. Such marked increase in efficiency, although obtained by theoretical considerations, serves to indicate very clearly the potentialities of compounding.

To get an idea of what change in performance would be expected in practice by compounding, the curves of Figs. 4 and 5 are computed on the basis of the assumptions stated previously. The improvement in performance is still appreciable in spite of the fact that the recovered

blowdown energy has to be decreased due to the unavoidable blowdown losses and turbine inefficiency. In investigating the actual performance the effect of the exhaust recovery losses on the increase in power and efficiency is rather compensated by the effect of the losses that have to be considered when an attempt is made to compute cylinder output from theoretical values. Fig. 4 and 5 show that the effect of compounding on performance is still interesting even when the blowdown recovery is not complete. It is interesting to notice from Fig. 4 that the percentage increase in power and efficiency due to compounding will

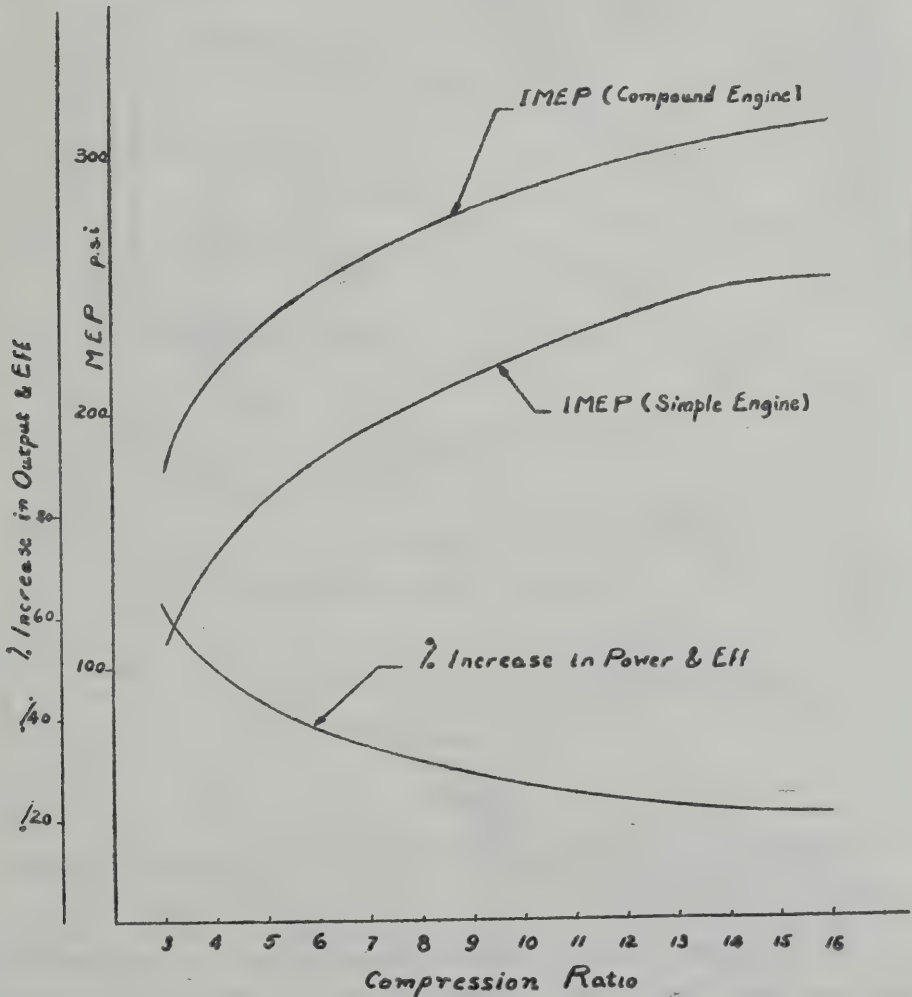


FIG. 2

vary from 62 to 22 between compression ratios 3 and 16 with 70 % recovery, which is approx. similar to what was obtained by the theoretical considerations presented in Fig. 2.

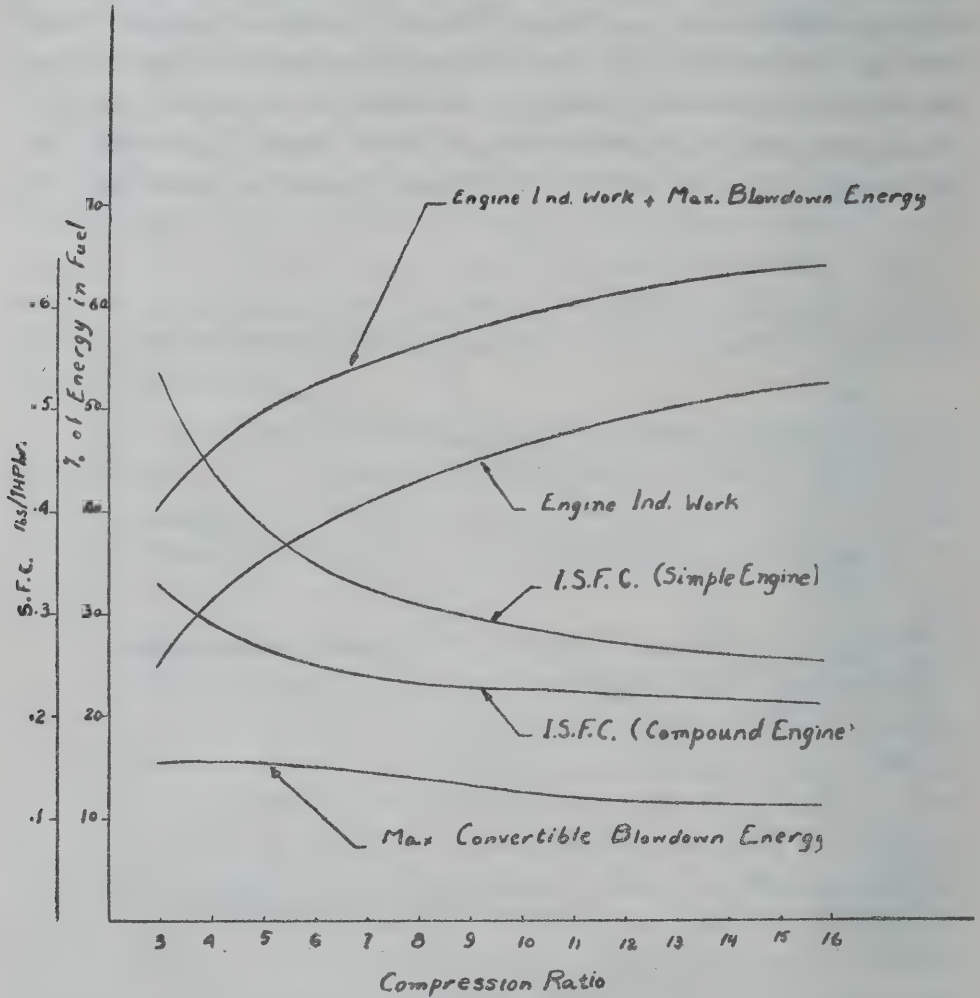


FIG. 3

Curves presented in Fig. 4 show that more returns are still obtained at lower compression ratios even if recovery is not complete

At compression ratio 6, the percentage increase in power and eff. decreases from 40 % with blowdown recovery of 70 % to 18.5 %, with 30 % recovery. This is still a marked improvement as Fig. 5 shows

that at a compression ratio of 6 the brake thermal efficiency of the engine is expected to increase from .26 to .36 with 70% blowdown recovery, and to .30 with 30% recovery. A brake thermal efficiency

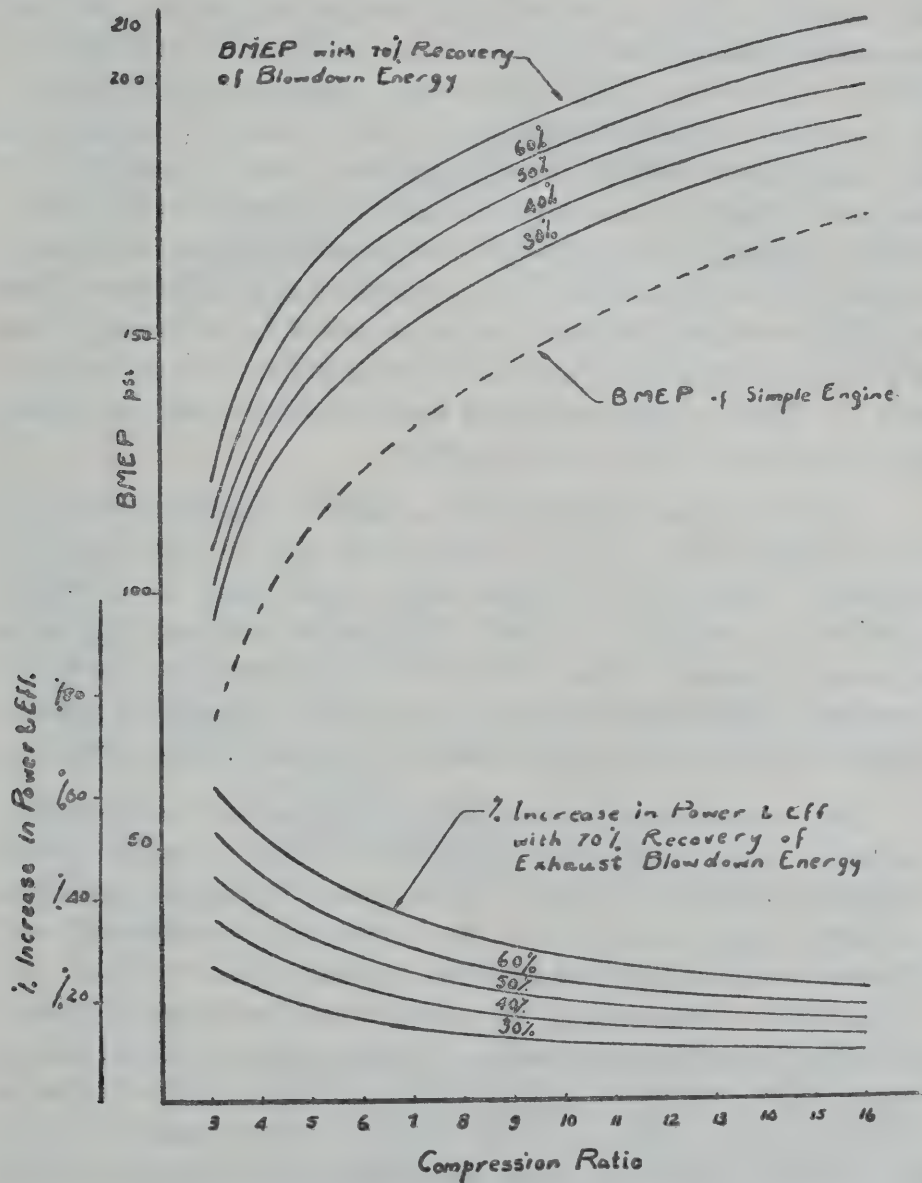


FIG. 4

of .3 is obtained with the Otto cycle engine at a compression ratio of 8.5. This clearly shows that if only 30 % of the convertible blowdown energy in the exhaust is recovered, the gain in power and efficiency will be equivalent to an increase in compression ratio from 6 to 8.5. This is an appreciable improvement since no one would underestimate the extensive effort in research and design required to raise the compression ratio of the engine to 8.5 and still use the same fuel as before. A low value of 30% for blowdown recovery is to be expected at the early stage of development of the compound engine, since information about exhaust ports and exhaust manifolds suitable for maximum blowdown recovery and about the turbine itself is unavailable at present. If more design data become available and exhaust recovery can be increased to 50 % , the brake thermal efficiency of an engine of a compression ratio 6 can be increased to .33 which is the efficiency to be expected from an Otto cycle engine of a compression ratio 12. Such a compression ratio seems far beyond reach at present due to the knock limits set by detonation.

Specific fuel consumption in lb per BHP hr. decreases from .515 for the simple engine of 6:1 compression ratio to .44 with 30% blowdown recovery at the same compression ratio. If blowdown recovery is improved to 50 and 70 %, specific fuel consumption is expected to decrease to .4 and .365 lb/BHP hr respectively. Such fuel economy is similar to what is obtained from compression ignition engines with higher compression ratios and higher cylinder pressures.

The ever increasing demand for higher output and efficiency by increasing engine compression ratio limited the minimum Octane number of the fuels to be used. Unfortunately high quality gasolines are rare in nature and have to be processed at considerable cost. Chemical compounds added generally increase cylinder deposits and engine wear. Compounding may open the field for the use of inferior fuels which are more abundant and cheaper to buy. A fuel having highest useful compression ratio of 4 is used in a simple engine at a brake thermal efficiency of .207. If 50 % of the blowdown energy is recovered, the efficiency of the compound engine will be .285 which can be obtained by a fuel of highest useful compression ratio of 7.4

in a simple engine. If the blowdown recovery is increased to 70% the efficiency increases to .315 which can be obtained by a fuel of highest useful compression ratio of 10 in a simple engine.

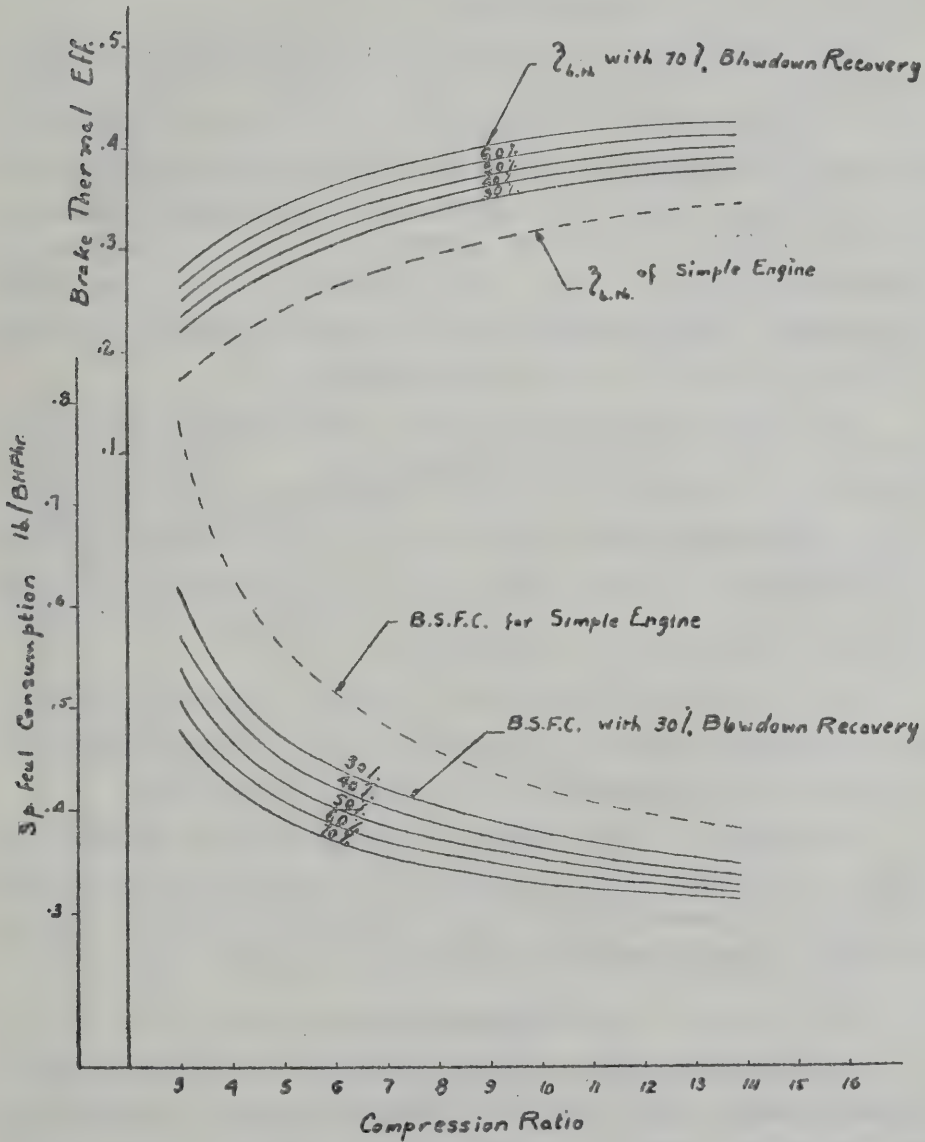


FIG. 5

PART-THROTTLE PERFORMANCE

The investigation of compound engine performance under part load conditions is presented in this paper for theoretical conditions only. It is believed such theoretical investigation is useful to study the characteristics of compounding.

Figs. 6 and 7 show that the gain in power and efficiency by compounding is reduced as throttling is increased. This is mainly due to the fact that as the pressure in the induction manifold is decreased, the temperature and pressure at the end of the expansion stroke decrease. There is a reduction in the percentage increase in power and efficiency from 39 to 18% when the inlet pressure is reduced from 14.7 to 8 psia. The gain is still quite appreciable at intermediate loads with increases of 31 and 25% at inlet pressures of 12 and 10 psia at the same compression ratio of 6.

Fig. 7 shows that under part load conditions compounding is still more effective at low compression ratios. The efficiency of the compound cycle behaves in a similar manner as that of the Otto cycle, as both efficiencies decrease with reduction in inlet pressure when the exhaust pressure is kept constant. Fig. 7 shows that the efficiency of the compound cycle at an inlet pressure of 8 psia is always higher than that of the Otto cycle at full throttle.

Fig. 9 present the indicated mean effective pressure of compound and Otto cycles at different inlet pressures which indicates that there is always an appreciable gain in output with compounding.

The indicated thermal efficiency and specific fuel consumption for both cycles for a wide range of part load operation are presented in Fig. 10. It is to be noticed that the drop in efficiency due to throttling is slightly steeper for the compound cycle.

CONCLUSION

The compound cycle proposed in this paper and shown in Fig. 1 is exactly similar to the Atkinson cycle which has been discarded and was impossible to utilize in a reciprocating engine due to the unequal expansion and compression strokes. In the proposed cycle the extra expansion will take place in the blowdown turbine.

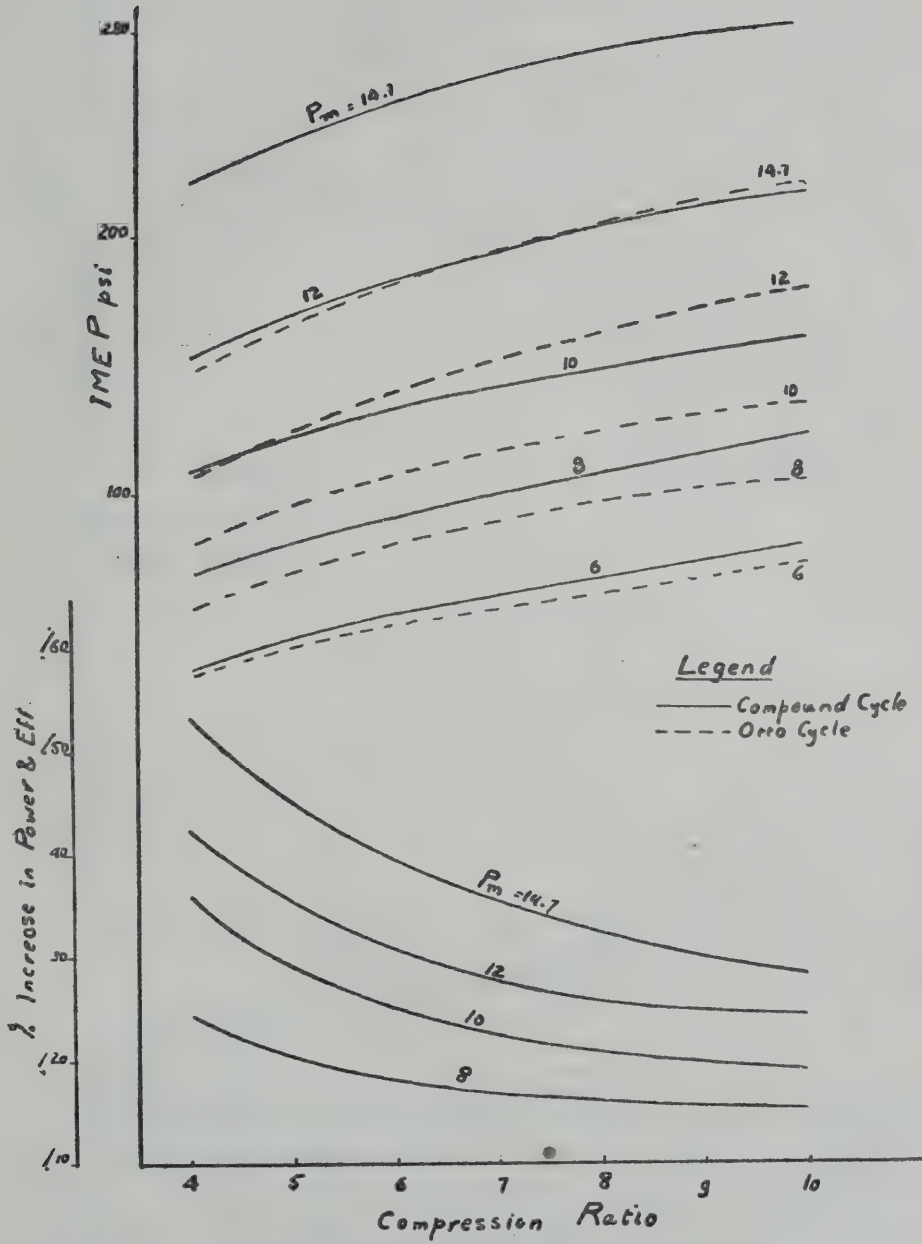


FIG. 6

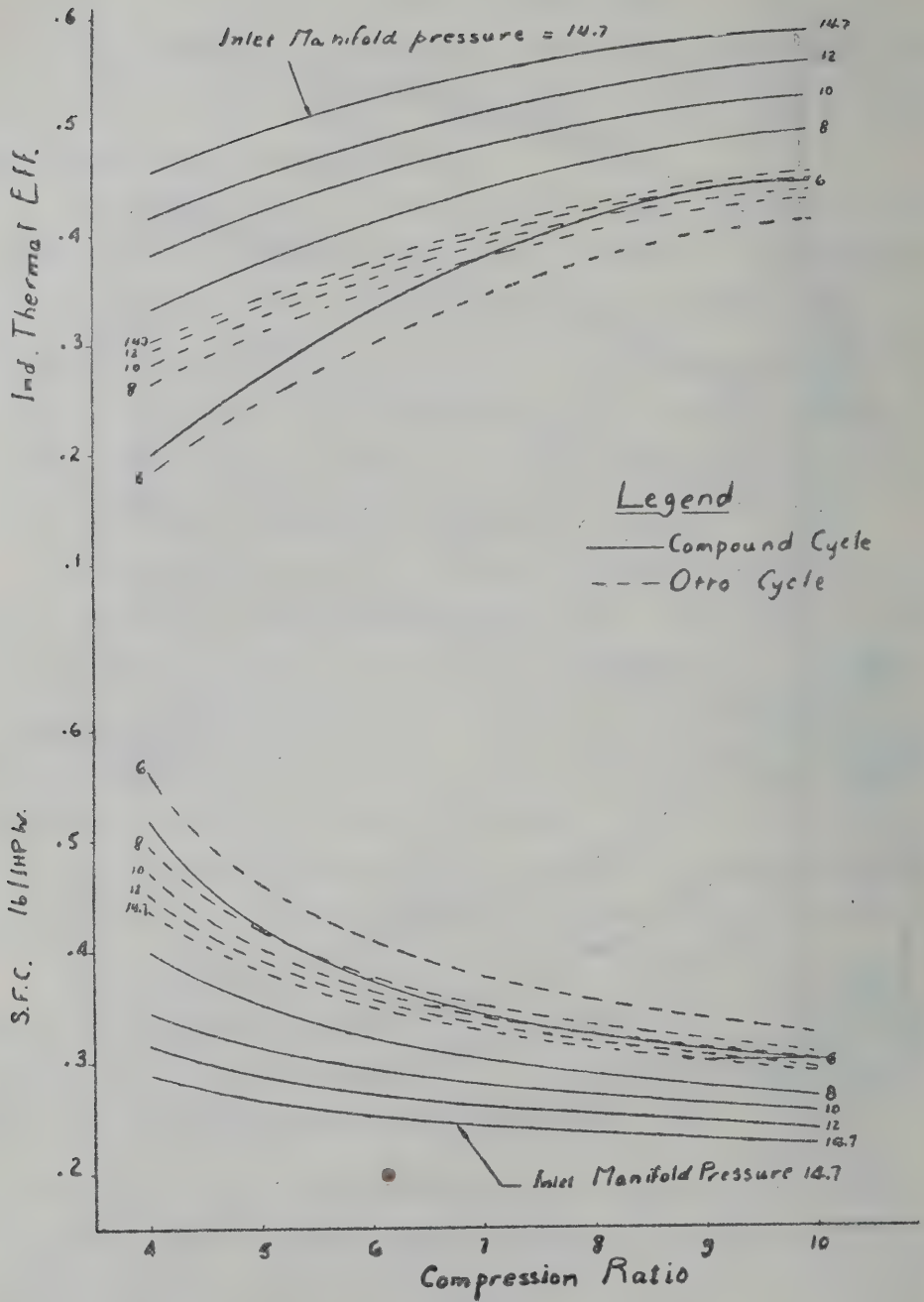


FIG. 7

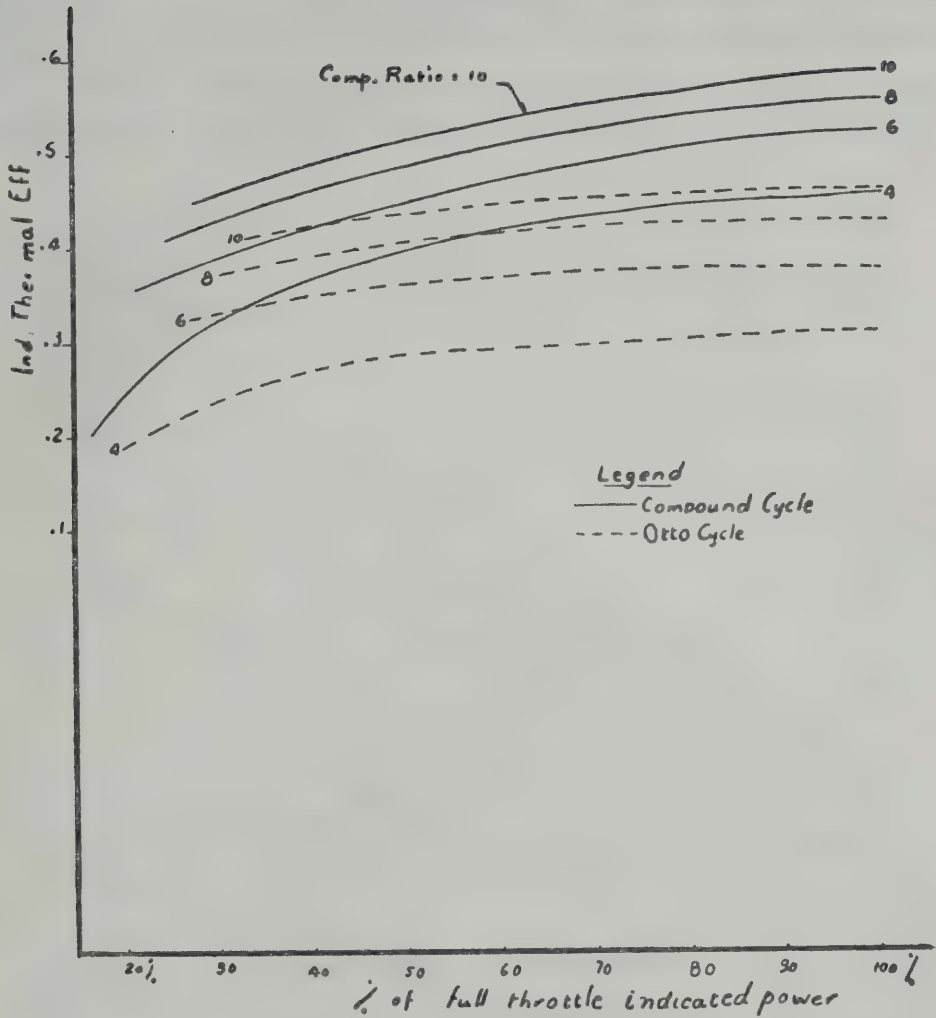


FIG. 8

Very little data is available at present dealing with the efficiency of the blowdown recovery process. The flow process is complicated and what improvement to be expected in energy conversion can be only obtained by actual tests conducted under a multitude of simulated operating conditions. This should not be discouraging as a considerable amount of research and testing and complete alterations in engine design have to be made at considerable cost to increase the

engine compression ratio slightly. No less effort and expenses have to be spent to raise the Octane Number of available fuels. Better fuel economy and more power output may be obtained if part of the effort given to raise engine compression ratio and the quality of gasoline fuels is directed towards the development of blowdown compounding

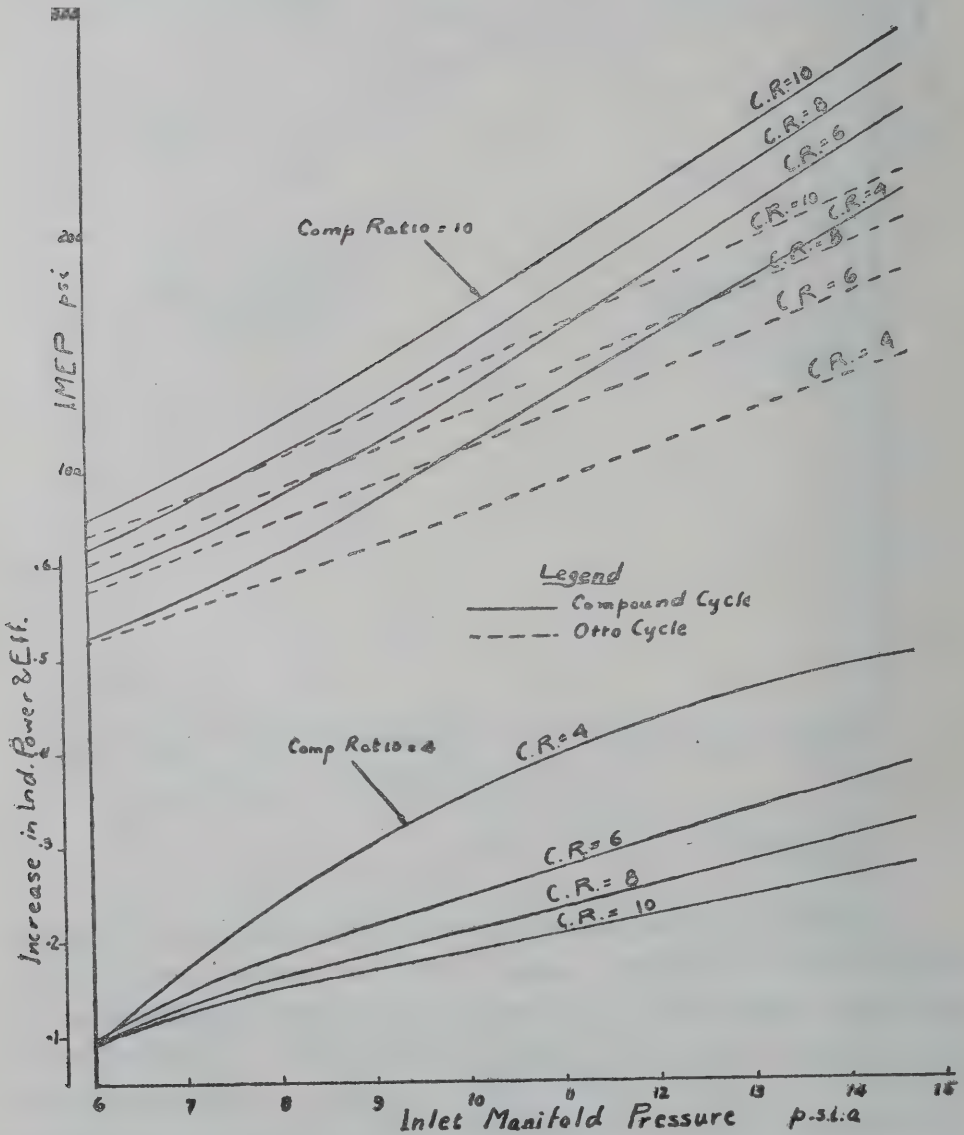


Fig. 9 •

of normally aspirated engines. These efforts may not exceed, if not less, than what have been spent to develop the new automatic transmissions.

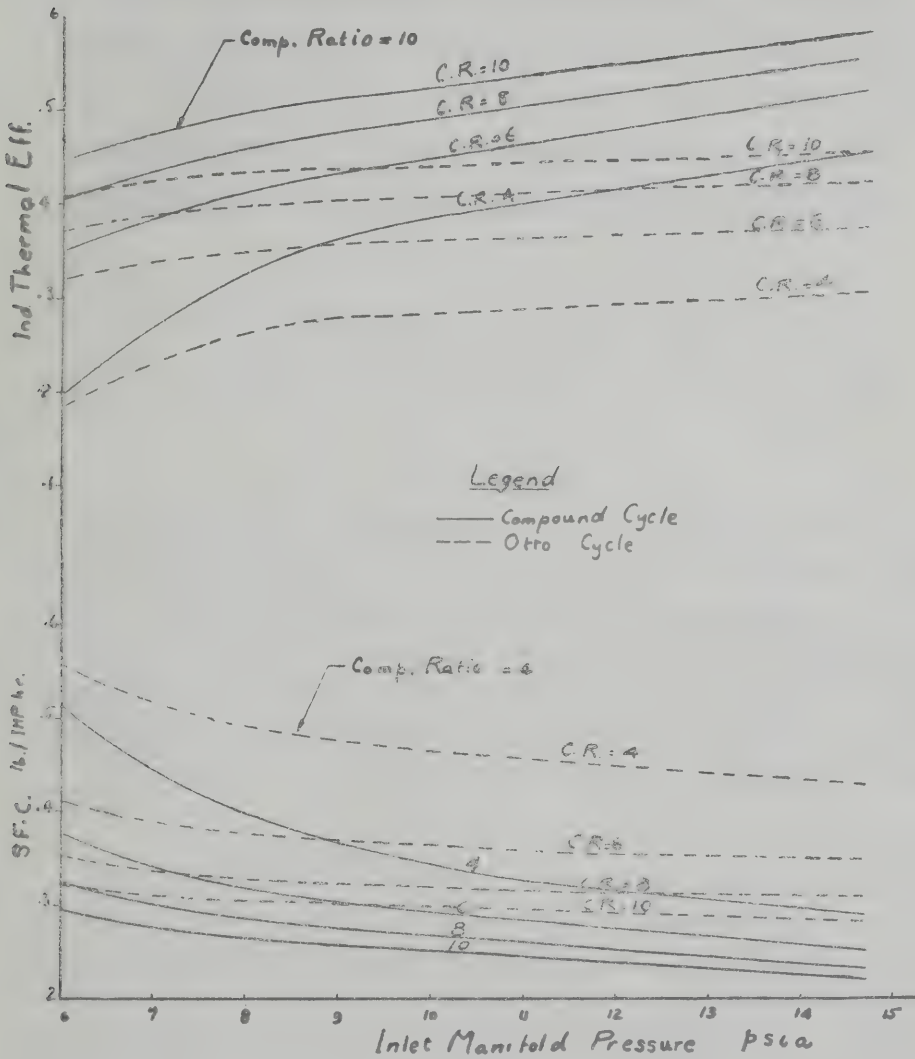


FIG. 10

Blowdown compounding may lead to aircooled engines of slightly lower compression ratios and of simpler design using cheaper gasoline and running at higher efficiency than present spark ignition engines.

A STUDY OF THE EFFECT OF GAS MEDIUM AND OIL TEMPERATURE ON LUBRICATING OIL DETERIORATION IN AN APPARATUS UNDER CONDITIONS AS THOSE PRESENT IN CRANKCASES OF SPLASH LUBRICATED INTERNAL COMBUSTION ENGINES.

BY

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INTRODUCTION

The use of fresh atmospheric air, as a scavenging medium, in crankcases of splash lubricated I. C. engines, is a recent practice adopted to inhibit oil-sludge formation, in crankcases of such engines, (R-1). It is meant by the work presented in this paper to study the effectiveness of this medium and of some other factors, which may be present in those crankcases, mainly the gas medium and the oil temperature, on the deterioration products, in an apparatus specially designed for this purpose.

Tests carried out in this work, show clearly that the oil stands stable for a certain time, before passing to breakdown due to the excessive accumulation of the deterioration products. This time which is referred to as the stability index, apart from being dependant on the operating temperature, was found also to be dependant on the type of gas medium present in the crankcase atmosphere. It was found also, that under different conditions of temperature, and having air as the circulating medium, the oil stands comparatively stable until a certain critical temperature is reached, after which it will start to show excessive amounts of deteriorating products.

Having realised the importance of this critical temperature, it is suggested, in this work, that the tests carried out for rating different oils, as regards their stability, should be limited up to the time showing their critical temperatures, irrespective of waiting for the actual occurrence of the oil breakdown. This meant the saving of a big effort as it was able to reach these temperatures in comparatively short duration tests.

It was also possible to show that unless the lubricating oil temperature is put under careful control, so as not to reach its critical temperature, under the different operating conditions, air which is used as the ventilating medium in practice, could be more detrimental to the oil than the other gases which may be present in the crankcase and being replaced by the air. This statement deals only with the deterioration products originating from the oil and not those coming from outside.

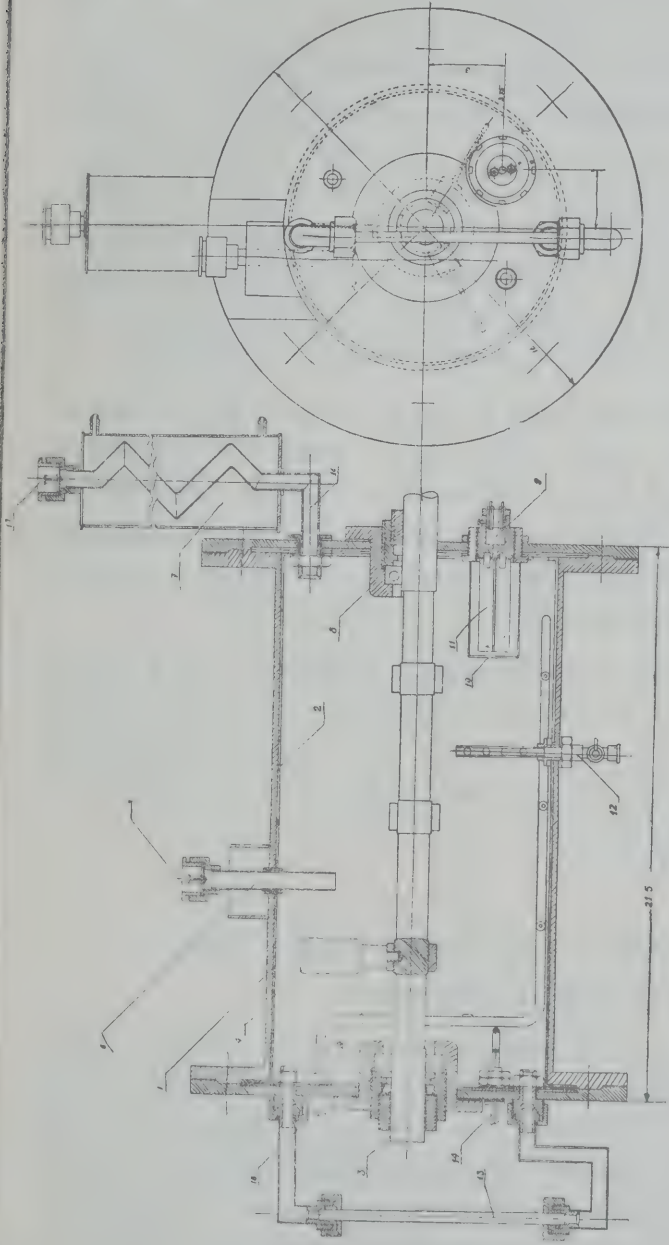
Oil deterioration, in splash lubricated I. C. engines, can be mainly looked upon to take place under two distinct different conditions. These are :

1. The rubbing action and heat while the oil lubricates the different engine moving parts.
2. The agitating action and heat in certain atmospheres, as exist in the crankcase of the engine.

In the current work, the second condition is only dealt with. The apparatus used in the investigation, is designed to duplicate, as near as possible, conditions existing in the crankcase of a splash lubricated engine. This can be considered as a reservoir of oil at a certain temperature, well agitated and kept in touch with different gases and metals, giving different catalytic oxidation effect. These factors are taken into consideration in the apparatus design which is shown clearly in Fig. 1.

The tests carried out in this work fall under the following two main groups of tests :

1. The first group of tests deals with the investigation of the effect of the different gas mediums on oil sludge, while the oil



APPARATUS DESIGN

DESIGNED BY
J. H. H. H. H.

1	Oil Inlet	11	Mica Core of Heating Wire
2	Condenser	12	Sampling Tube
3	Crank Stainer	13	Level Gauge
4	Three Heating Elements	14	Thermometer
5	Now Return Valve	15	Oil Retainer
6	Internal Liner	16	Gases Outlet
7	Crank Stainer	17	Non Return Valve
8	Three Heating Elements	18	Gases Inlet
9	Heater Insert		
10	Oil Retainer		
11	Mica Core of Heating Wire		
12	Sampling Tube		
13	Level Gauge		
14	Thermometer		
15	Oil Retainer		
16	Gases Outlet		
17	Non Return Valve		
18	Gases Inlet		

FIG. 1.—DETAILED DESIGN OF THE APPARATUS

1. Main Iron Casing.
2. Internal Liner.
3. Crank Stainer.
4. Three Plades.
5. Now Return Valve.
6. Oil Inlet.
7. Condenser.
8. Main Bearing.
9. Three Heating Elements.
10. Heater Insert.
11. Mica Core of Heating Wire.
12. Sampling Tube.
13. Level Gauge.
14. Thermometer.
15. Oil Retainer.
16. Gases outlet.
17. Non Return Valve.
18. Gases Inlet.

temperature is kept constant at 280 deg. F. This temperature is the one adopted in the 36 hours engine test, as the crankcase oil temperature, when rating different oils for stability (R-2). The oil tested in this group is a straight mineral naphthenic base oil A, and the tests are carried out under a constant charge of the oil, of about 4.5 litres. This amount can be considered as a mean value in practice. The gas mediums chosen for these tests are: Air, Oxygen, Carbon dioxide, Nitrogen, and different types of exhaust gases of known analysis. The tests are carried out at intervals of 8 hours daily run, at the end of which a sample is taken, while the oil is still under the test conditions and analysed. The same procedure is carried out for several intervals and the rate of sludge formation is recorded for every condition.

2. The second group of series of tests deals with the investigation of the temperature factor. The same procedure is adopted as above, but with varying the oil temperature. Air is chosen as the circulating gas medium in this series of tests. The same naphthenic base oil as above is again tested together with the testing of another straight mineral paraffinic base oil B.

In all the previous tests, the apparatus is lined with a copper liner which is claimed to have the most catalytic effect as regards oil deterioration (R-4). All other different parts inside the apparatus and are in contact with the oil, are nickel plated to render them inert as regards oil deterioration.

I.—EFFECT OF GAS ON OIL DETERIORATION UNDER THE SAME TEMPERATURE.

Fig. 2, gives the rate of increase of both the petroleum ether insolubles and the chloroform solubles with the test time, when testing the naphthenic oil A. It shows that the rate of increase of these products is comparatively small for a certain time depending on the test condition of the percentage presence of oxygen in the circulating gas medium, before it shoots up suddenly, The point at which the rate of increase of these sludge measurements starts being very high, no doubt indicates the time after which the oil under test becomes

unstable. This meant that the oil has attained a critical state of deterioration and accordingly the time of each test is limited to that giving this sudden increase in the rate of increase of these deterioration products. In the current work, the time taken by the oil to show a sudden rush of its sludge insolubles is going to be referred to as the stability index of the tested oil under the specified test conditions.

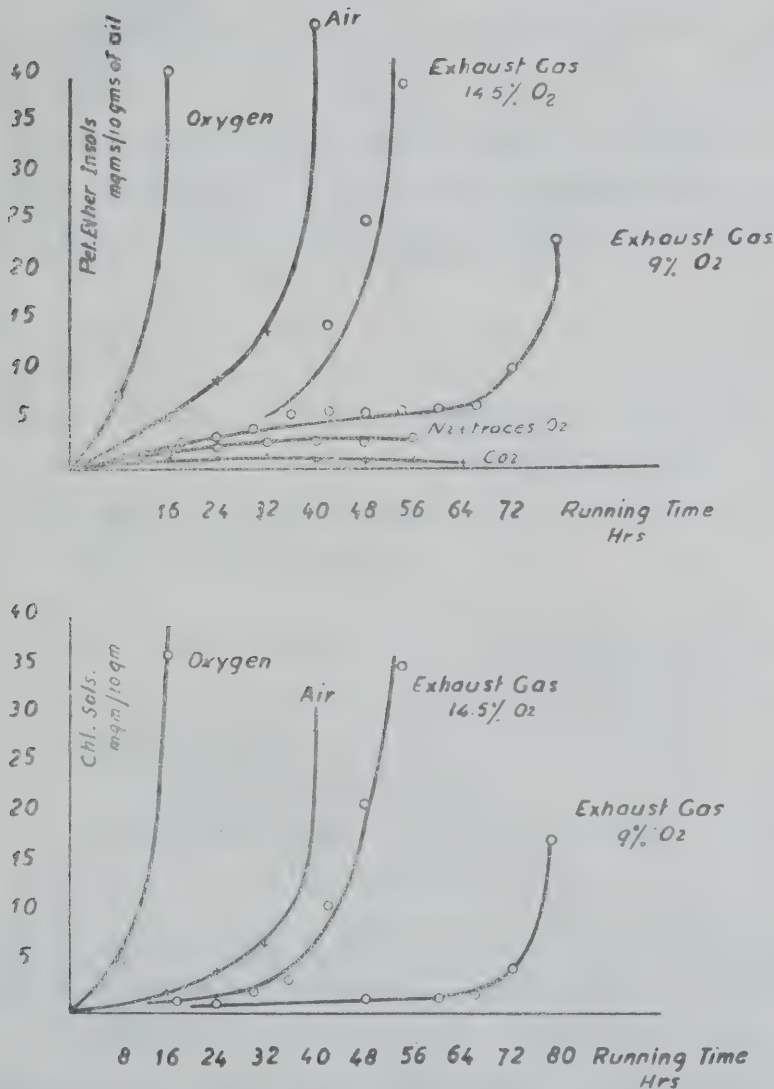


FIG. 2.—Change of Pet-Ether Insols. and Chl. Sols. with Test Time. Test Temp. 280°F—Naph. Oil.

When there is only traces of oxygen, or no oxygen at all, the rate of increase of these deterioration products, remains constant and comparatively small for a very long time, or it may tend to decrease. This is shown by curves of N_2 and CO_2 in the same figures.

The rate of increase of the neutralization number in all the above tests is given in Fig. 3, which shows that in all cases, where there is enough amount of oxygen, the rate of increase follows exactly a behaviour opposite to the one given by the oil insolubles. The rate of increase starts very high at the beginning, but decreases gradually until a maximum is reached. It may be of interest to notice that under the same test conditions, the time at which the maximum neutralization number is reached coincides nearly with the time at which the rate of increase of the oil deterioration products increase suddenly.

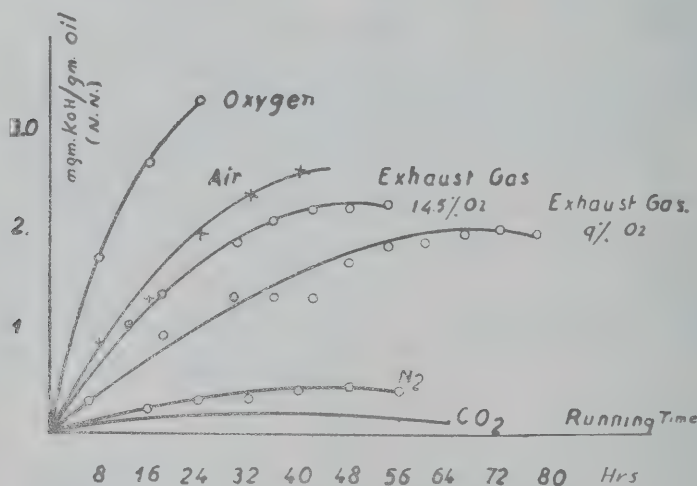


FIG. 3.—Neutralization Number with Test Time.
Test Temp. 280°F.

Fig. 4, shows that the viscosity of the oil under all conditions increases at a constant rate. This rate increases with the increase of the oxygen percentage in the atmosphere in the apparatus.

Thus, it may be concluded, that under the test temperature, oxidation is the main factor in the oil deterioration, in fact these tests show that the presence of air, (21 % O_2), is more harmful than the

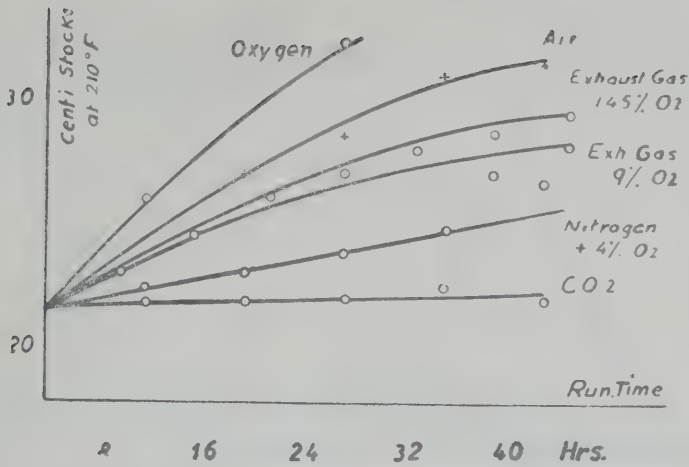


FIG. 4.—Viscosity with Time.

exhaust gases, (9 % , 14 % O₂). When there is no oxygen present, the change in the sludge measurements is practically nill. This statement can be more realized by the study of Fig. 7, which gives the relation between the stability index of the tested oil under the defferent operating conditions, and the amount of oxygen in the circulating gas medium.

II.—EFFECT OF THE CHANGE OF THE OIL TEMPERATURE ON OIL DETERIORATION WITH AIR AS THE CIRCULATING GAS MEDIUM.

Fig. 5, shows the effect of the increase of the test temperature on the sludge measurements when testing the naphtenic oil with air as the circulating gas medium. A study of this figure shows that the effect of increasing the oil temperature becomes more as with high temperatures. In fact, an increase of temperature from 220 deg. F. to 265 deg. F., gives a much smaller change in the increase of the sludge measurements, than an increase from 265 deg. F. to 280 deg. F. Thus, the increase in the oil temperature has a slight effect on oil stability, under comparative lower temperatures, until a certain critical temperature is reached, after which any further increase in temperature has a marked effect on the stability of the oil.

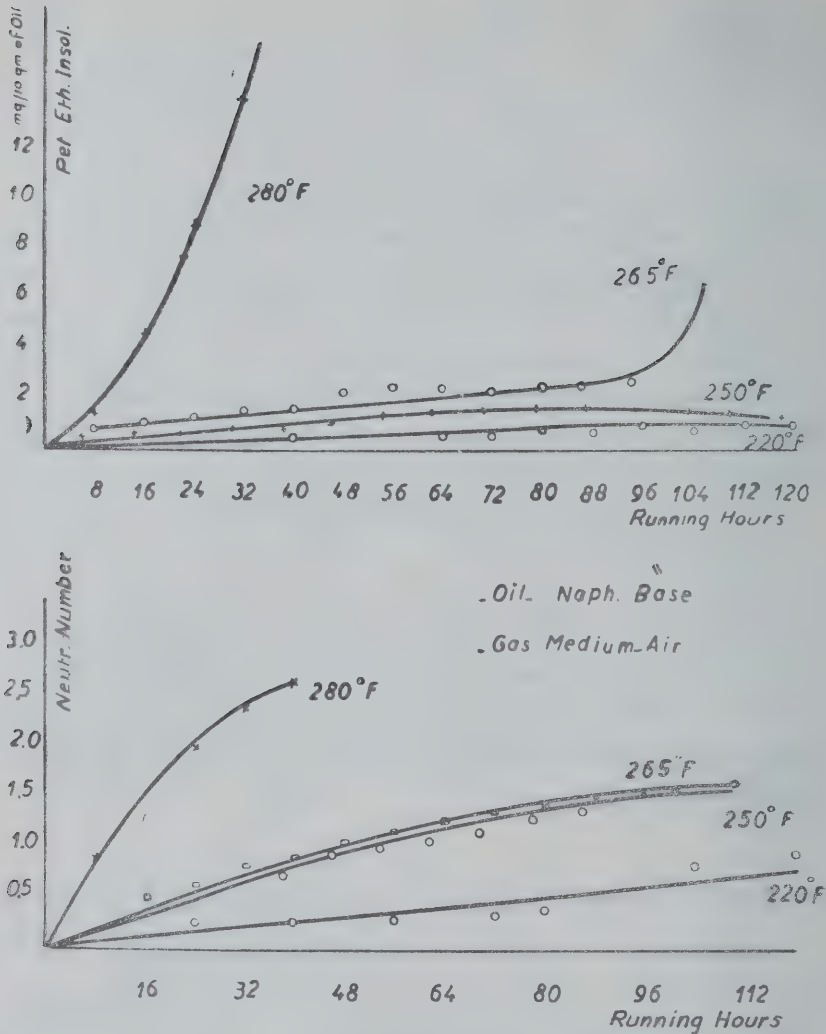


FIG. 5.—Effect of Oil Temperature on Sludge measurements.

Fig. 6, gives the change in the value of the petroleum ether insolubles with temperature, for certain test times of 40, and 32 hours, and shows that the increase in those insolubles is very small up to temperature of 260 deg. F., and after which their rate of increase with the increase of oil temperature becomes very high. This temperature could possibly indicate a critical stage of temperature for the tested oil under the specified test condition and beyond which, the oil is comparatively stable. Fig. 6, shows also that a short duration test, of

about 40 hours, is quite enough to be able to determine the critical temperature of the tested oil, irrespective of carrying out long run tests, as given in Fig. 5, in trying to search for the breakdown temperature.

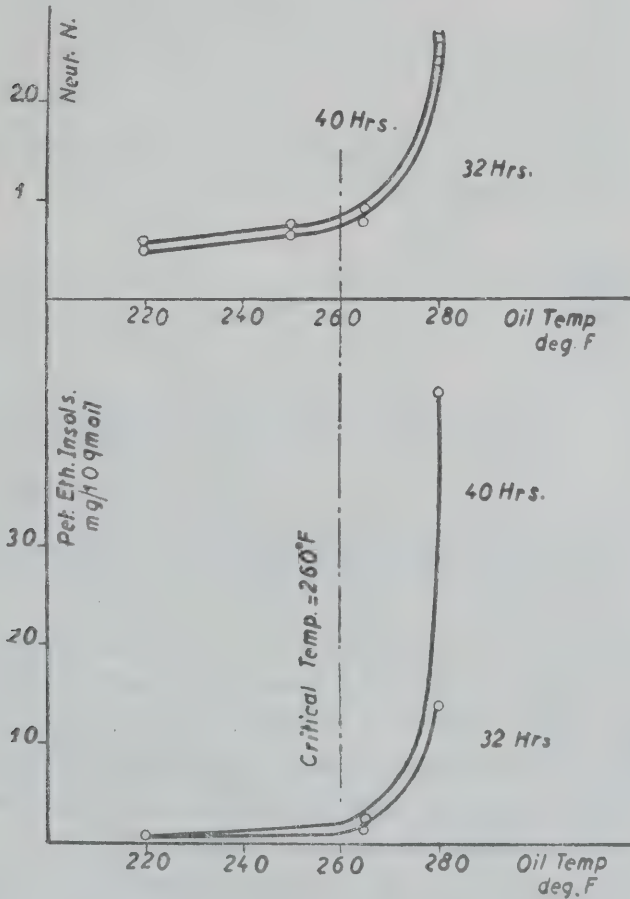


FIG. 6.—Short run Tests Show the Oil Critical Temperature.

Although this work is not intended to study the behaviour of different oils or to make a comparison between them, but rather to establish testing methods, yet it was thought useful to determine the critical temperature of a paraffinic base oil (*B*) when tested under a short duration testing procedure. Fig. 9, gives the results of such

tests and shows clearly that the paraffinic base oil, although is run under different conditions of test time of 24, 32, and 40 hours, yet they all agreed in giving a constant critical temperature of 315 deg. F.

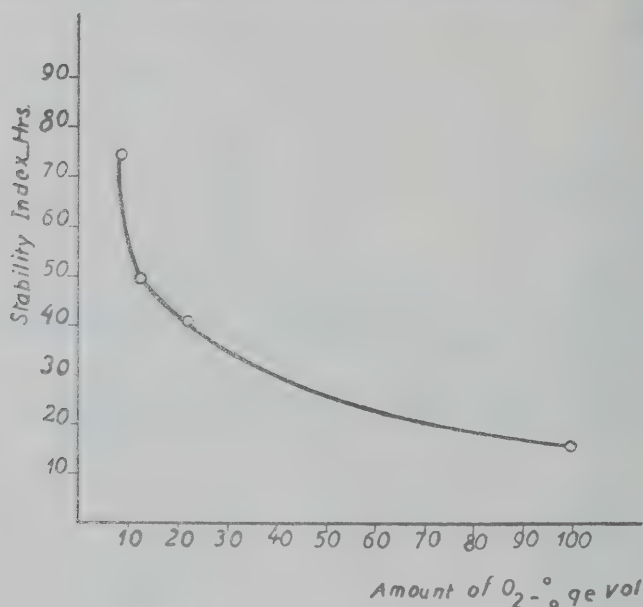


FIG. 7.—Effect of Amount of Oxygen in ventilating Medium on Stability Index For Naph. Oil.

The naphthenic oil (A), by giving a critical temperature of 260 deg. F. (127.5 deg. C.), which is much less than that given by the paraffinic oil (B) which is 315 deg. F. (157.5 deg. C.), proves the fact given by other investigators, in stating that lubricating oils of the paraffinic origin are more stable from the point of view of the petroleum ether insolubles formation (R-5, 6, 7).

In a recent work carried out by the Author on a diesel engine, (R-8,), it was possible also to demonstrate the critical stage of temperature, which always precedes the oil breakdown. When testing two oils of the same above origin for ring sticking, the paraffinic base oil gave a critical temperature of 135 deg. C., while the naphthenic base oil gave a much lower temperature of 105 deg. C. It may be of some interest to notice that the difference in the critical temperatures for the two base oils in the two testing procedures is nearly the same and equals 30 deg. C.

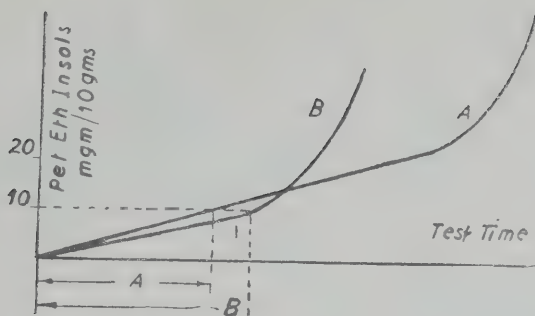


FIG. 8.—Rating Oils According To Their Stability Index and the Indiana Testing Standard.

Moreover, comparing the results of the rate of increase of the petroleum ether insolubles with test time, obtained in this work, with those obtained in a similar work carried out when testing different oils under the Indiana Testing Procedure, and on engine tests, Fig. 10, (R-9), it can be noticed that the behaviour of the oils as indicated by the apparatus developed by the Author, conform more with those obtained by the engine tests. They both start by a slight and gradual increase in the insolubles, which is followed by a sudden rush period. The results obtained by the Indiana Testing Procedure, which is a bench test used for testing the stability of different oils, (R-9), does not show any increase in the insolubles for some time, and then the high rate of increase starts suddenly. This, together with the help of the previous discussion, may indicate that the apparatus developed in this work, gives testing conditions nearer to those obtained in an engine, than those obtained in the Indiana Test. The truth in this statement remains to be seen when further work is carried out to correlate the results obtained by this apparatus with those obtained in actual engine tests.

Thus this work may lead to the fact that for testing the stability of an oil, two factors have to be determined. These are :

1. The critical temperature of the oil.
2. The stability index of the oil at temperatures above the critical temperature.

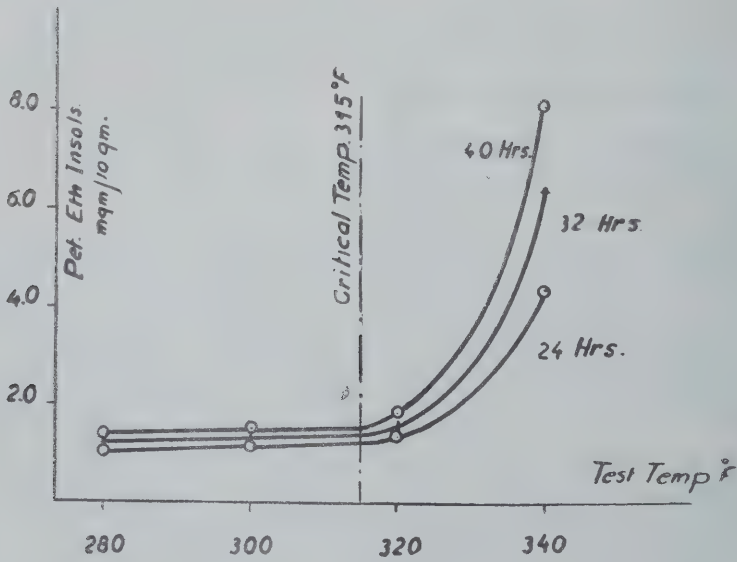
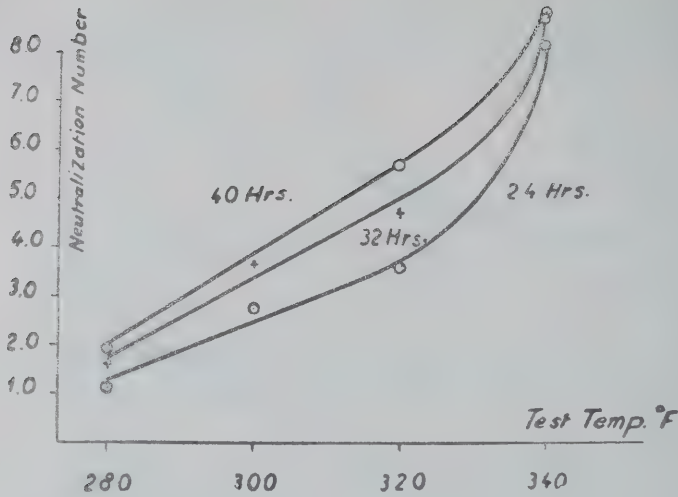


FIG. 9.—Effect of changing Test Temp. on Sludge Measurement for Paraffinic Oil B.

This method may be more accurate than the rating of different oils, as regards their stability, by comparing the time taken to form 10 mgms. of insolubles per 10 gms. of oil, which is the method adopted for rating different oils in the Indiana Testing Procedure. The

following example can help in clarifying this statement, Supposing that two oils A, and B. When tested for stability gave results as shown in Fig 8. If the time to form 10 mgms. of petroleum ether insolubles is taken as an indicator of oil stability, the oil B will be rated as more stable than oil A, while in fact oil A is more stable as it gives higher stability index.

TYPICAL SLUDGING TESTS.
CURVES OF OILS RUN IN THE ENGINE
AND IN THE INDIANA OXIDATION TEST.

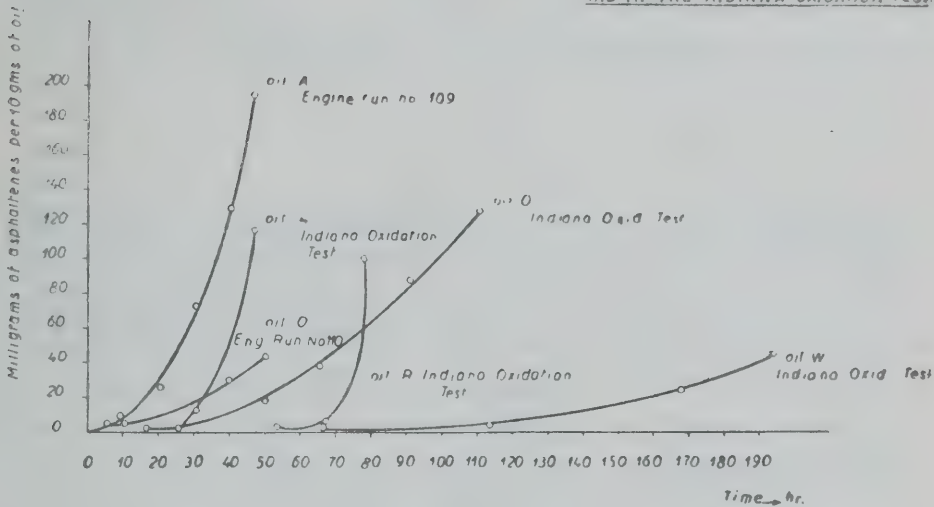


FIG. 10

This work shows also that the most important factor in the deterioration of oils at high temperatures is oil oxidation. At such temperatures which should be within the range of the critical temperature of the tested oil, the stability index decreases with the increase of oxygen in the crankcase gas medium. This may throw some doubt on the subject of crankcase ventilation, as an effective procedure to inhibit sludge formation, in crankcases of I. C. engines under the different operating conditions. The results obtained in this work indicate that using air as the ventilating medium in the crankcases of such engines may be of no use, and perhaps more harmful, if the working temperature of the lubricating oil in the crankcase exceeds its critical temperature. This could be the condition of operation of such type

of engines, when working in hot stmospheres. However, this statement can not be taken for the moment as definite, since there are many other factors involved, but this indicates that the problem of crankcase ventilation should have clearer studies than it has been given up to now.

In conclusion, the Author feels that the results obtained in this work are not complete, and further work on this subject is already planned. The results obtained so far, however, show clearly the way to be followed and is an indication of the necessity of further work on the subject generally, and it is felt that this work has pointed out to the way to be followed in further investigation.

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A PROPOSED SCHEME FOR THE POWER CONTROL OF CONSTANT SPEED GAS TURBINES

BY

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AND

S. A. FARAG, B.Sc., M.Sc., D.I.C., Ph.D.

INTRODUCTION

We no more refer nowadays to the gas-turbine as the "prime-mover of the future". The term has become an under estimation. The gas-turbine has already established a firm stand in many applications and is offering a strong competition with conventional types of prime-movers in many others. In the range of small and medium Electrical Power generation, for example, the gas turbine has many advantages over the steam plant. The greatest of these lies in the absence of a water circuit with all the problems connected with it such as feed water treatment. etc. Also, relatively low operating pressure, simplicity of design, low first cost, compactness, quick readiness for being put on duty and insignificant maintenance costs are only some of the many other advantages of the gas-turbine over its competitor, the steam turbine in this field of application (1).

So far, the use of gas-turbines in this field was almost restricted to units answering peak-load requirements and running in conjunction with already existing Electrical Power stations of the steam or hydraulic type. However independent all-load gas turbine power stations are being built nowadays in ever increasing numbers (2). This gives the question of part-load performance of a gas-turbine unit used for Electric Power generation a greater importance than it has so far assumed.

With this point in mind and with the belief that the gas-turbine is well suited for Electrical Power generation in some isolated parts of our country, we have taken over the present study of the Power control of Gas-turbine units used for this purpose.

GAS-TURBINE PART-LOAD CONSIDERATIONS

The type of variation of efficiency with output demanded by different duties of a gas-turbine varies considerably. As regards the type of duty under consideration namely all-load electrical power generation, a gas-turbine will, it is presumed, require as small a variation in efficiency as it is possible to obtain for changes in output down to low relative powers of say 20 % design power.

The possibility of good part-load efficiency for the desired working range, promised by any particular component arrangement is of little value, however, if the arrangement encounters major practical difficulties in operating over this range. These difficulties (excluding those of mechanical nature) are :

1. Surging of the compressor or compressors.
2. Over-heating of one or other of the turbines.
3. Over-speeding of one or other of the rotors.
4. Over-heating of the H. E. Tubes.

Of these difficulties surging presents the most impossible limit and it seems likely that this condition must be avoided at all costs. Because of stress difficulties both over-speeding of any component and over-heating of a turbine are likely to have adverse effects on the life of an engine and should in consequence be avoided if at all possible (3).

METHODS OF POWER CONTROL

Although the possible arrangements of components in a gas turbine installation run to a great number, those used for electrical power generation can be put under two main general classifications (4) :

1. Installations in which the turbine which drives the Electrical generator also supplies power for the drive of one or more compressors.

2. Installations whose gas turbine comprises two separate parts (with or without reheat between them) one which drives only the compressor, while the other supplies the effective power to the generator.

In any case, the generator which gives a constant frequency electricity supply must run at constant speed at all loads. It thus follows that in the first classification the compressor speed must also be constant at all loads while in the second classification the speed of the compressor set can (or must) be reduced during periods of part loads. The power-turbine in this case will run independently at constant speed.

Regulation of load in each of the two types of engines is carried on as follows :

(1) In an installation of the first type where the compressor speed is constant at every load, the air quantity would be very nearly constant. Matching of components (compressor and turbine) according to the characteristics of both, necessitates, however, a decrease in both pressure ratio and max. temp. as the load is reduced. This is readily shown by the "ellipse law" expression (5).

$$\frac{m \sqrt{T_m}}{P_m} = K \sqrt{1 - \frac{1}{R^2}}$$

conventionally used for denoting the characteristics of a multistage turbine and where

m = mass flow.

T_m = max. temp.

P_m = max. pressure.

R = Pressure Ratio

and K = a constant

2. In an installation of the second type where the compressor speed drops with reduction in load, the air quantity compressed as well as the turbine pressure ratio drop with load. The control is thus achieved mainly by varying the air quantity while the maximum

temperature is maintained approximately constant. Below a certain load, however, it is usually necessary to lower the maximum temperature as well in order that the heat exchanger tubing is not exposed to the high exhaust temperature resulting from the lower turbine pressure ratio at these low loads (4).

COMPARISON BETWEEN METHODS OF CONTROL

Figure 1 reproduced here from reference 4 shows the influence of these two types of control on the efficiency of the unit. It is clear that the first type of control namely that with a constant speed compressor, gives a lower relative part load efficiency.

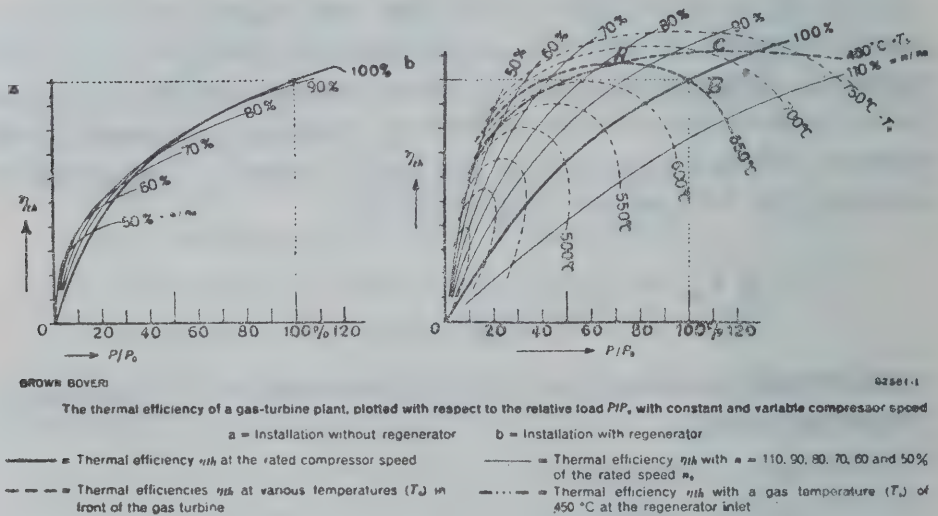


FIG. 1

The figure also shows that the second type of control where the compressor speed is varied with load, gives quite a favourable part load efficiency. Below a certain load (Point A) however, the maximum temperature (curve B) has to be lowered due to the limitation of the temperature at the inlet to the Heat Exchanger (curve C).

Fig. 2 shows a typical overall characteristics of a high duty axial compressor⁽⁶⁾. According to method (1) of control a lower load at constant compressor speed would mean a shifting of the working point towards the steep part of the constant speed line and hence a tendency towards choking and the troubles connected with it. The range of load variations, therefore, had to be very narrow. Moreover, any appreciable variation of load in this case would mean a big drop in the compressor efficiency. Scheme (2) of Power control gives no doubt a much higher relative part load efficiency. However, a compressor designed for certain optimum conditions of speed and efficiency will, when run at other speeds, be exposed to stage mismatching. Quoting A.D.S. Carter (Ref. 7, Page 5-33): "As most compressors have to operate over a speed range, some mismatching will occur. In examining the performance under such conditions, it will be assumed that there is some speed at which the compressor is matched. This will correspond to the point of maximum polytropic efficiency. A stage-by-stage calculation, based on mean values for this flow, would show a fairly constant incidence relative to optimum at the mean diameter, *i.e.* the optimum flow coefficient is maintained at each stage. Such conditions cannot be maintained for all stages other than at the matching point. For example, at lower speeds less work will be done by the first stage. Consequently, the density will be less than design at entry to the second, and once the flow area is constant the fluid velocity will be higher. This results in even less work being done by the second stage and a higher velocity at entry to the third and so on through the compressor.

The variation of incidence on successive blade rows will tend to become modified as shown in Fig. 3. The incidence increases towards stall on the L.P. stages and decreases towards choke on the H.P. stages as the speed is reduced until some speed is reached at which these occur simultaneously. Below this speed the compressor can only operate with one or more stage permanently stalled. The compressor itself does not surge, due to the stabilising effect of the H.P. stages".

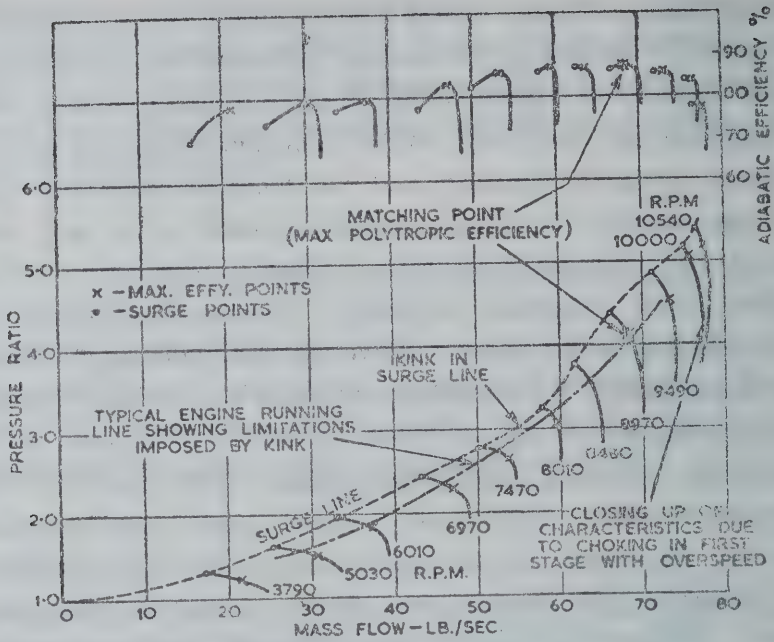


FIG. 2.—Typical Overall Characteristics of a High Duty Axial Compressor
(Results from A. R. Howell, *Proc. I. Mech. E.*, Vol. 163, p. 243.)

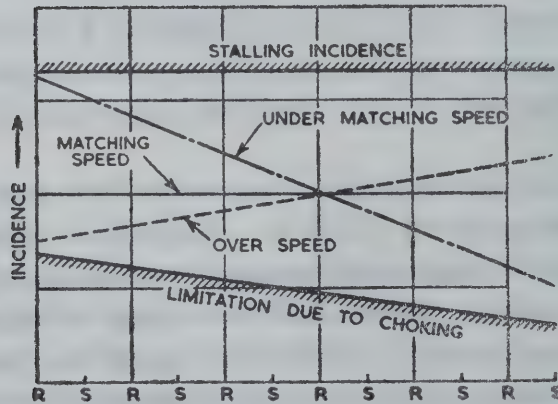


FIG. 3.—Variation of Incidence through High Duty Compressor at Various Speeds

In units using a high pressure ratio, stage compression (with or without intercooling) is necessary. Compressors mounted on same shaft present in this case a great difficulty in matching at off-design points particularly when intercooling is used (Fig. 4)(8). At lower speeds, there is a tendency for the low pressure compressor to surge and for the high pressure compressor to choke.

It is obvious from the above discussion that a great deal of refinement in compressor design would be of no value if the compressor in an all-load unit is going to be exposed to the above mentioned troubles at off-design points. As is well known, a compressor designer find it necessary to resort to a great number of compromises in design : at the design point, *i.e.* point of maximum Polytropic efficiency, optimum conditions are chosen with regard to incidences, etc. with the result that at other working points aerodynamic considerations such as angles of incidence etc. would not be optimum. However, since the compressor has to operate over a speed range, the designer would have to sacrifice optimum considerations in order to minimise stage mismatching at off-design points.

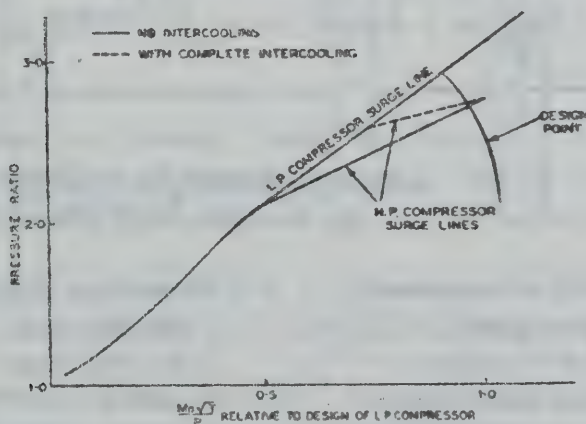
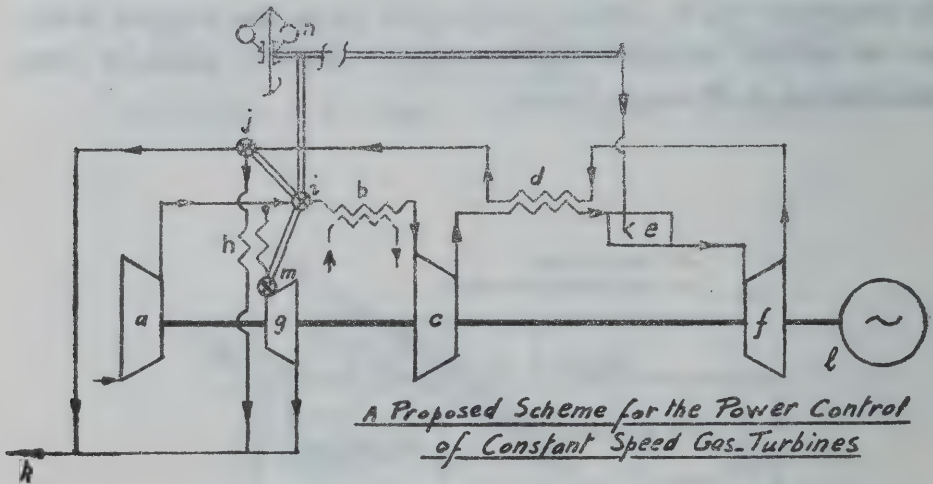


FIG. 4.—Skeleton Characteristic of Two Compressors in Series on the Same Shaft

THE PROPOSED SCHEME FOR POWER CONTROL

It is obvious from the above discussion of the methods of power control that as far as the compressor in a gas-turbine unit is concerned, it would be of great advantage if some method could be devised by means of which the power of the unit is varied while the compressor operates constantly at the same point on its characteristics at all loads.

Fig. 5 shows a new scheme proposed by the authors for the power control of a gas-turbine unit in which the above condition is fulfilled. At full load, the operating conditions are exactly the same as those in a conventional gas-turbine unit. The air compressed in the first stage compressor *a* after passing through the intercooler *b* is compressed to the maximum pressure in the second stage compressor *c*. The air passes then through the heat exchanger *d*, the combustion



- | | |
|--------------------------------|-------------------------------------|
| <i>a</i> = L.P. Air Compressor | <i>h</i> = Auxiliary Heat-Exchanger |
| <i>b</i> = Intercooler | <i>i</i> = Throttle Valve |
| <i>c</i> = H.P. Air Compressor | <i>j</i> = By-Pass Valve |
| <i>d</i> = Main Heat-Exchanger | <i>k</i> = Stack |
| <i>e</i> = Combustion Chamber | <i>l</i> = Electric Generator |
| <i>f</i> = Gas-Turbine | <i>m</i> = Adjustable Air Passages |
| <i>g</i> = Air-Turbine | <i>n</i> = Governor |

Fig. 5

chamber *e* and finally is expanded in the main turbine *f*. The three main components are on the same shaft and thus run at the same speed.

At Part Loads, the speed has to remain the same irrespective of the load. The same amount of air, irrespective of load is compressed in the first stage compressor and to the same pressure ratio as at full load. Now, a part of the air compressed in the first stage compressor is led through a throttling arrangement *i* and goes through the same sequence of events as before. The remaining part of air compressed in compressor *a* passes through an auxiliary heat exchanger *h* before being expanded back through nearly the same pressure ratio of the first stage compressor in an air turbine *g* mounted on the main shaft of the unit and having adjustable inlet passages *m*. The throttle *i* and the adjustable inlet passages *m* are controlled by the governor *n* simultaneously. In this case, the net power developed by the unit will be simply.

$$\begin{aligned}\text{Net Power} &= \text{Power generated by main turbine } f \\ &+ \text{Power generated by air turbine } g \\ &- \text{Power absorbed by the two compressors } a \text{ and } c.\end{aligned}$$

The main features of operation at part load can be summarised as follows :

1. Except for pressure losses, the expansion ratio in the air turbine is the same as the compression ratio in the first stage compressor.

2. Since air is considered to behave as a perfect gas, the throttling process will take place with the following results (°) :

(a) Pressure after throttling P_i is proportional to the mass flow through the throttle.

(b) The total temperature of air remains the same after as before throttling.

3. The first stage compressor operates at all loads at exactly the same point on its characteristics since the dimensionless parimeters

$$\frac{N}{\sqrt{T_1}}, m \frac{\sqrt{T_1}}{P_1} \text{ and consequently } \frac{P_2}{P_1} \text{ and } \eta_c \text{ do not change where}$$

m = Mass flow

N = speed of rotation

T_1 = Inlet Temperature

P_1 = Inlet Pressure

P_2/P_1 = Pressure ratio.

and η_c = Insentropic efficiency.

4. The main turbine f characteristics is expressed by :

$$\frac{M_f \sqrt{T_{\max}}}{P_f} = K \sqrt{1 - \frac{1}{R^2}}$$

where m_f = mass flow through turbine.

$T_{\max}$ = turbine Inlet Temp. = T .

P_f = Turbine Inlet Pressure.

K = a constant.

and R = Turbine Pressure ratio.

In order to maintain as well the same operating point on the characteristics of the second stage compressor, P_f has to be controlled such that the pressure ratio for this compressor is kept the same at all loads. Thus its dimensionless parameters $N/\sqrt{T_i}$ and $m\sqrt{T_i}/P_i$ will also obviously remain the same at all loads where :

T_i = Inlet temp. to second stage compressor (Temperature of air after intercooling).

P_i = Inlet Pressure to second stage compressor (Pressure after intercooling).

This is achieved by varying $T_{\max}$, the turbine pressure ratio R being deduced from the above expression (ellipse law). $T_{\max}$ is controllable through the fuel flow into the combustion chamber.

5. Preliminary investigations of the new scheme led to the possibility of exploiting the heat contained in the exhaust gases leaving the main heat exchanger in heating the air expanded in the air turbine g , thereby gaining extra useful work. An auxiliary heat exchanger h is used for this purpose, the amount of exhaust gases passing through it being controllable through a valve i coupled together with the throttle i and the mechanism for the adjustable passages m to the governor n of the gas-turbine unit.

Fig 6 shows a T.S. diagram for a gas-turbine unit working in accordance with the presented scheme.

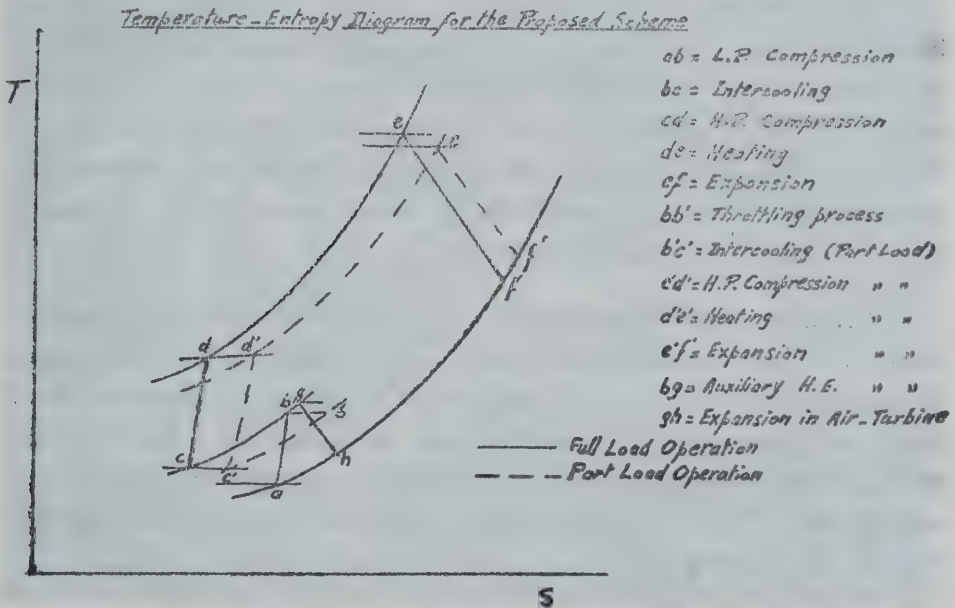


FIG. 6

THE AIR-TURBINE

Air turbines have already been in use for the purpose of output control of turbo-compressors in coal-fields, shipyards, mines, etc. (10)

They are then generally referred to as recuperation turbines. The recuperation turbine was first introduced by the firm Brown Boveri and Co. Ltd. Baden (Switzerland). An extract from an article in the Firm's Review Nos. 11/12 Vol. XXX of November/December 1943 is given hereafter :

"In 1935, the automatically-operating recuperation turbine as incorporated in the latest isotherm-type compressors was introduced. This turbine converts the energy in the excess compressed air—which with atmospheric discharge control is completely wasted—into useful power which it immediately reapplies to the compressor shaft. This method of control developed by the Company gives efficient operation and immunity from surges for all loads from overload practically down to zero capacity".

The success of this method led to its introduction by other firms for similar purposes (¹²). An account of its use in the regulation of axial blowers for blast-furnaces and in steel works is given in an article in the Sulzer Technical Review No. 3 (1955) from which certain useful data relating to this type of turbine was borrowed for the purpose of carrying out the present investigation. Extracts from this article is given hereafter :

"... axial blowers are fitted for this purpose with a recuperation turbine. This system, which may also be adopted for centrifugal blowers, consists in running the blower at maximum delivery at all times and expanding any surplus air in a turbine. The turbine wheel, which is coupled to the blower shaft thus recovers most of the energy used for the compression of the surplus air and thereby relieves the driving motor".

"The new recuperation turbine has adjustable guide blades. The profiled guide blades form nozzles in which the medium is expanded, so that it enters the rotor blades at high velocity. An alteration in the pitch of the blades changes the nozzle cross-section and with it the air throughput. The result is continuous volume regulation without throttling losses, at least down to very nearly

closed blade positions. The rotatable blades also modify the direction of flow in accordance with the volume, in this way favourably affecting the efficiency at all loads. Finally, the rotor always operates with full admission, so that optimum working conditions are ensured in the diffuser".

Figs. (7, 8 and 9) are reproduced from the above article and show the features of such a turbine which we believe to be quite suitable for incorporation in the proposed scheme presented here for the power control of constant-speed gas-turbines. Fig. 10 shows the variation of adiabatic efficiency of the air turbine with throughput i.e. the rate of mass flow through it.

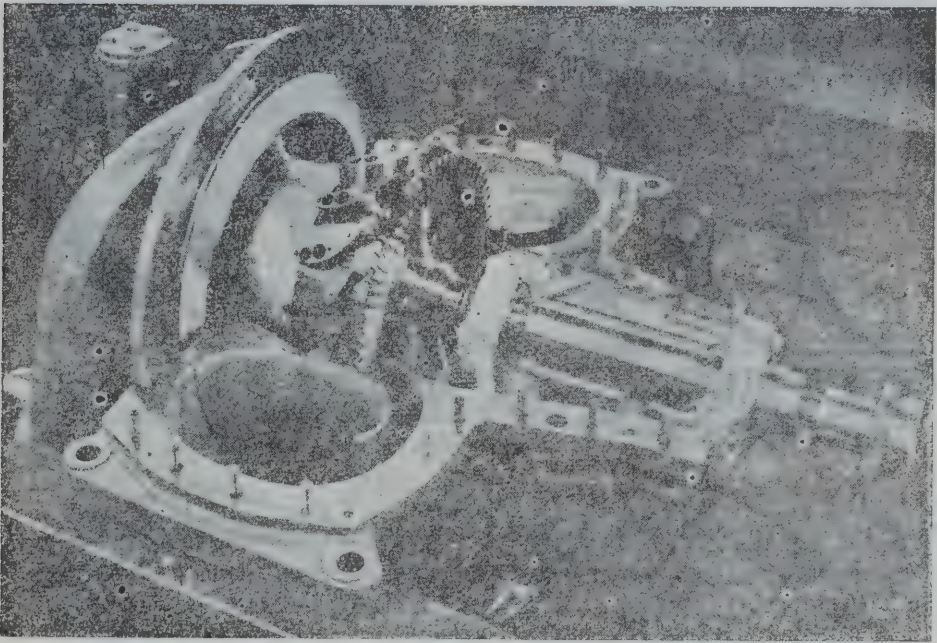


FIG. 7.—New Sulzer recuperation turbine for regulating the delivery of axial blowers running at constant speed

In the course of carrying out the present investigation, trials were made for finding out the best position of the air turbine in the

arrangement of components of the gas-turbine unit in so far as it gives a high part-load efficiency relative to that at full load. The air-turbine was first incorporated in the arrangement so as to follow the completion of the compression process and the study was carried out for each of the two cases where the compression is done either in a single compressor or in two stages with intercooling. In each of these two cases the air turbine expansion pressure ratio is nominally the same as the overall compression pressure ratio. The second way in which the air-turbine was incorporated in the arrangement is where it followed the 1st stage compression and before the intercooler in a two stage compression arrangement. The latter alternative gave the best part load efficiency relative to that at full load. The air-turbine expansion pressure ratio in this case is equal to the compression pressure ratio of the first stage compressor.

The optimum pressure ratio for the L.P. compressor in the arrangement adopted here at a given part load is that which gives

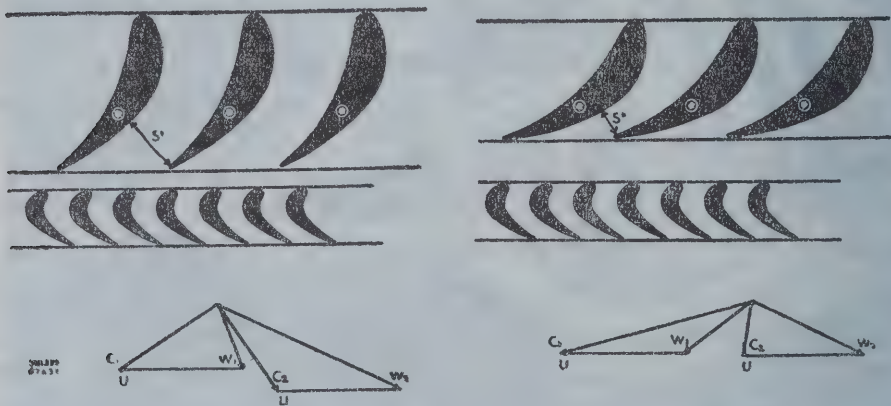
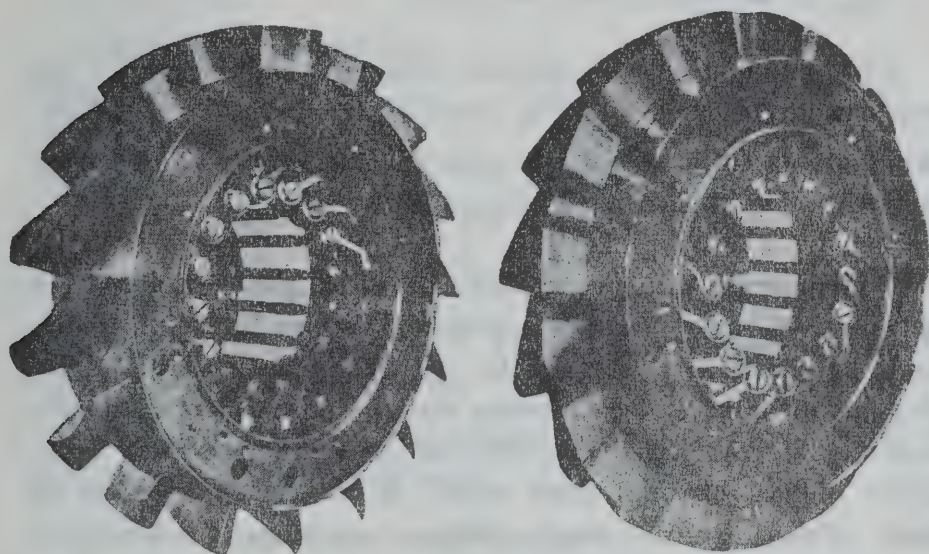


FIG. 8.—Blading diagram of the Sulzer recuperation turbine with adjustable guide blades when the guide blades are fully open (left), the nozzle cross-section S' is available for the expansion of the air. When the blades are closed (right), the nozzle cross-section is reduced to S'' and the quantity of air admitted is diminished in almost the same proportion without any throttling losses being caused. The velocity vector diagrams for the two blade positions are shown at the bottom.

a minimum value for the net work done in the compression process. The latter is given by

(work done in L.P. compressor + work done in H.P. compressor-work done in air turbine).



Completely open blade position

Completely closed blade position

FIG 9.—Guide-blade wheel of a Sulzer recuperation turbine the blade adjusting mechanism is enclosed in an oiltight and dustproof casing and connected to the forced lubrication system of the blower. It thus requires no special maintenance and is subject to hardly any mechanical wear.

In the appendix at the end of this paper, it is shown how a formula was obtained for calculating this optimum pressure ratio.

WORKED EXAMPLE OF A GAS-TURBINE UNIT OPERATING ACCORDING TO THE PROPOSED SCHEME

In order to explore the possibilities and merits of the proposed scheme, a detailed example of a constant-speed gas-turbine unit operating in accordance with it is worked here-after.

The unit has the following data :

$$\text{Ambient conditions} \quad \left\{ \begin{array}{l} \text{Pressure 1 atm. abs.} \\ \text{Temp. } 15^{\circ}\text{C} = 288^{\circ}\text{K} \end{array} \right.$$

$$T_{\text{max.}} \text{ at full load} = 700^{\circ}\text{C} = 973^{\circ}\text{K.}$$

The optimum overall pressure ratio of compression for the arrangement considered here (Fig. 5) and for the chosen $T_{\text{max.}} = 700^{\circ}\text{C}$ (taking into account practical values for the pressure losses, intercooler effectiveness etc...) is found to be equal to 6 : 1 ⁽¹¹⁾.

Preliminary investigations showed that for proper matching of components, the maximum temperature *i.e.* Turbine Inlet temperature had to be varied only very slightly with variations in load. With this point in mind and considering that the unit will operate more frequently at part loads, the full load pressure ratio was taken slightly higher than the above value. It was taken therefore as 6.5 : 1. This is to ensure nearly optimum conditions at a convenient part load rather than at full load.

Isentropic Efficiency of compressors : 88 % .

This is slightly on the higher side than is frequently met with in practice⁽¹²⁾. However, we consider it to be allowable since one of the results of the present investigation (as will be explained later) is that producing compressors of as high an efficiency as possible is worth all the trouble given to it when such a compressor is intended for use in units operating according to the present scheme.

Isentropic Efficiency of Power Turbine . 88 %

Combustion Chamber Efficiency 90 %

Thermal Ratio (Effectiveness) of main
and auxiliary heat exchangers 80 %

PRESSURE LOSSES
Including ducting etc.

L.P. compressor Intake	1 %	of Local total head pressure.
Intercooler	4.5 %	„
Haat Exchanger (air side)	4 %	„
„ „ (gas side)	5 %	„
Combination Chamber	2 %	„
Auxiliary Heat Exchanger (air side)	4 %	„
„ „ „ (gas side)	4 %	„

Cooling air Bleed: 2 % of compressed air, taken from the delivery of the H.P. compressor.

Auxiliary drives and mechanical losses are considered to be equal to 3 % of compressors work.

The Optimum pressure ratio for the L.P. compressor, as shown by the deduced formula in the Appendix depends among other factors on the mass flow rate through the throttle. The value of this optimum pressure ratio at a part load corresponding to a mass flow rate of 90 % of full load mass flow was calculated and adopted as the constant pressure ratio of that compressor at all loads. The choice of this condition of operation namely using the optimum ratio at 0.9 mass flow rate was based on the fact given in preliminary investigation that this particular mass flow gave a relative Power output of nearly 75 % of full load. It is considered that this relative load would be that at which the unit will run most of the time.

As contemplated, the value for the overall compression Pressure ratio at 0.9 mass flow rate which was used in the formula given in the appendix was found to be very nearly 6.0 : 1.

This is the real optimum value of the overall pressure ratio for the arrangement ⁽¹¹⁾.

The following tables gives the results of calculations for this example:

TABULATED RESULTS OF A WORKED EXAMPLE OF A GAS-TURBINE
UNIT WORKING ON THE PROPOSED SCHEME

Mass Flow in Power Turbine Percentage of Full load flow. M_f	70	75	80	85	90	95	100
Mass Flow in Air Turbine Percentage of Full load Flow $100-M_f$	30	52	20	20	10	5	0
Adiabatic Efficiency of Air Turbine (from curve) % Fig 10.	83	85	81	73	65	50	—
Power turbine Entry Pressure P_f atm. abs	4.24	4.54	4.84	5.15	5.45	5.75	6.05
Power Turbine Pressure Ratio R . . .	4.02	4.32	4.60	4.88	5.17	5.46	5.75
Power Turbine Entry temp. T_m °K . .	940.5	949.8	955.9	960.6	966.0	970.0	973.0
H.P. Compressor Delivery Pressure atm. abs.	4.50	4.83	5.14	5.47	5.78	6.12	6.44
Intercooler Entry Pressure P_t atm. abs.	1.83	1.96	2.09	2.22	2.35	2.48	2.61
H.P. Compressor Entry Press. atm. abs.	1.74	1.87	1.99	2.12	2.25	2.37	2.49
H.P. Compressor Press Ratio . . .	←			2.58			→
H.P. Compressor Delivery temp °K .	←			416.5			→
H.P. Compressor Work C.H.U./lb. of air compressed	18.31	19.61	20.92	22.23	23.54	24.84	26.15
Total work of Compressors C.H.U./lb. of air compressed	44.85	46.20	47.40	48.80	50.20	51.55	52.75
Power turbine work C.H.U./lb. of air compressed	46.15	51.80	57.65	64.00	70.10	76.30	82.10
Power Turbine Exhaust temp °K . .	699.0	696.8	691.9	685.1	681.0	676.5	672.0
Combustion Chamber Entry temp °K .	642.5	640.7	636.8	631.4	628.1	624.5	620.9

TABULATED RESULTS OF A WORKED EXAMPLE OF A GAS-TURBINE
UNIT WORKING ON THE PROBOSKD SCHEME (Con.)

Heat Exchanger outlet temp °K . . .	484.5	483.8	482.9	480.8	480.5	478.8	477.5
Air turbine entry temp °K	465.9	465.3	464.6	462.9	462.7	461.3	460.3
Air Turbine Work C.H.U./lb. of air compressed	7.60	6.45	4.92	3.32	1.97	0.76	—
Net work C.H.U./lb. of air compressed.	8.9	12.05	15.17	18.52	21.87	25.51	29.35
Net work percentage of full load . . .	30.45	41.20	51.80	63.30	74.90	87.20	100
Heat Added G.H.U./lb. of air compressed	56.80	62.90	69.25	76.20	82.50	89.50	96.31
Thermal Efficiency %	15.65	19.18	21.90	24.30	26.50	28.50	30.35
Thermal Efficiency Percent of Full load thermal eff %	61.75	63.40	72.50	80.30	87.65	94.20	100

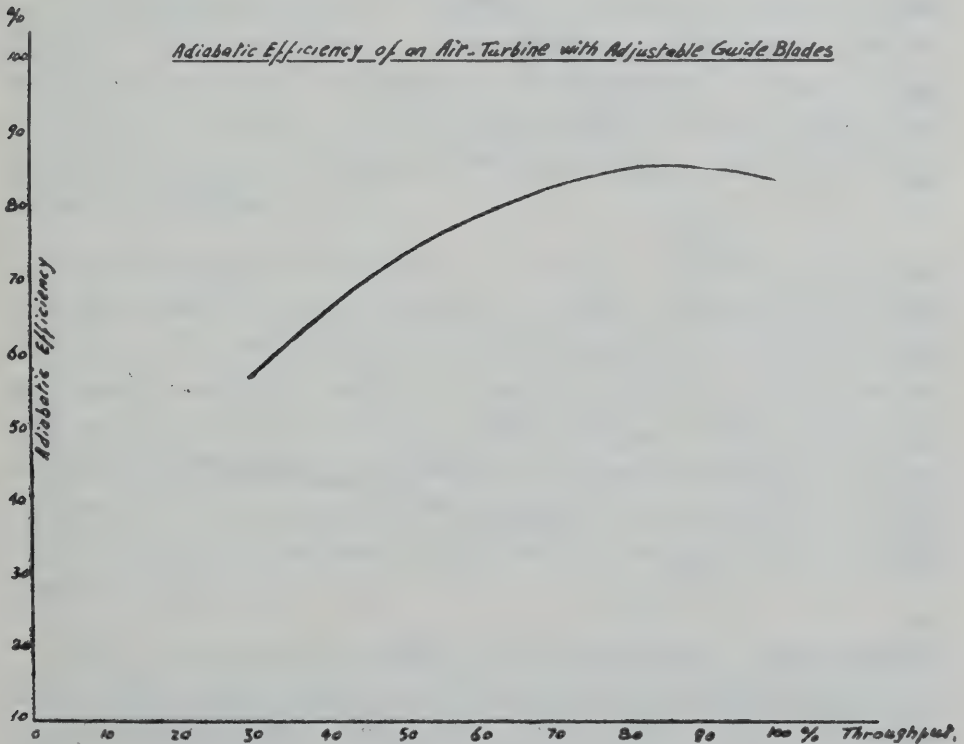


FIG. 10

The curves in Fig. 11 show the variation with respect to load of : both the thermal efficiency and the efficiency relative to that at full load for the unit of the worked example.

CONCLUSION

In the light of results obtained for the worked example, general conclusions can be obtained regarding the working limits, possibilities

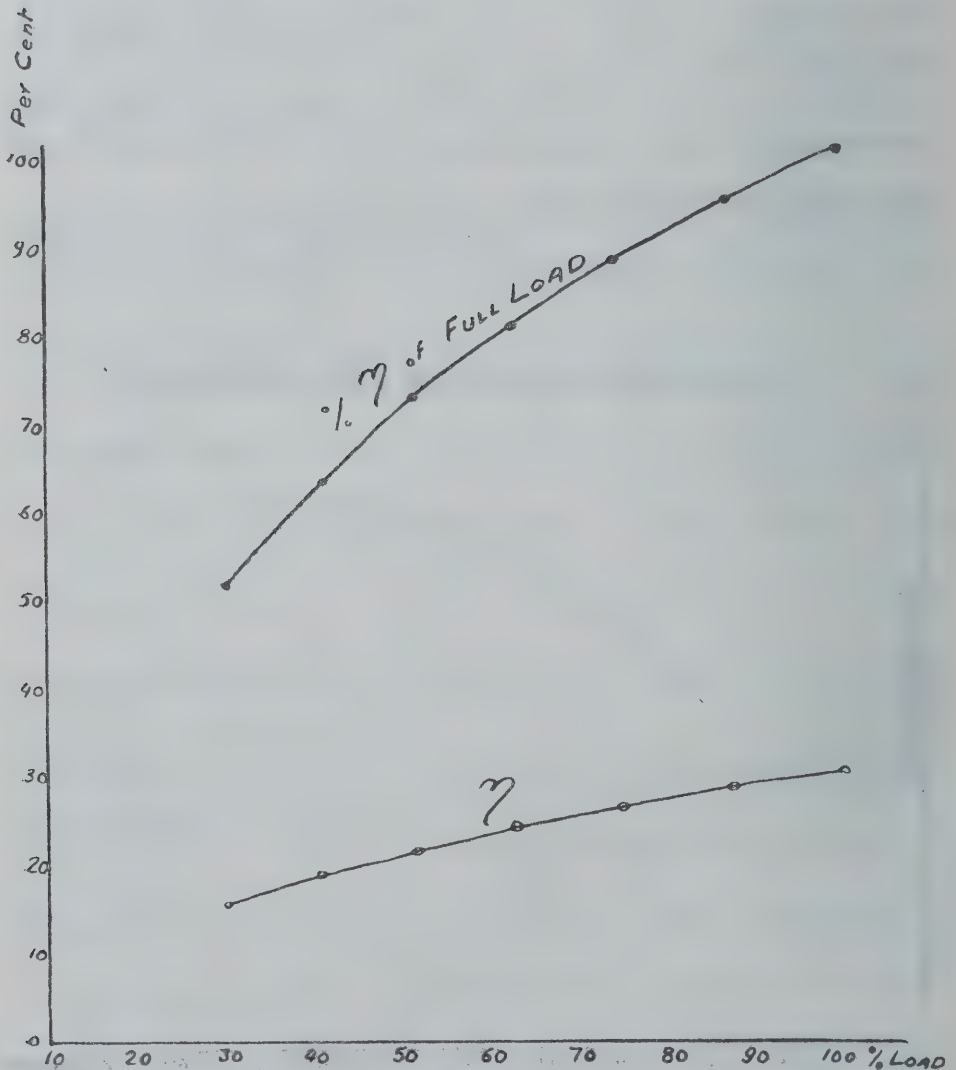


FIG. 11

and advantages of the proposed scheme presented in this paper. These conclusions can be listed as follows :

1. The new method proposed here for the power control of gas-turbine units results in part load efficiencies which lie between those given by the conventional methods of power control discussed before. This is made clear by the comparison between the efficiency curve for the new scheme and those for the other two schemes Fig. 1.

2. An obvious advantage of the new scheme is the compactness of the unit working in accordance with it. It would be a single shaft unit requiring the minimum of space, maintainance etc. Moreover, it would be of a higher efficiency at part loads than a unit of constant speed compressor of the conventional type even though the latter may be comparable with it in compactness, space requirements, maintainance, etc.

3. In comparison with a unit with variable speed compressor, the proposed unit may give a lower part load efficiency. However, efficiency is only one of the factors to be considered. From the point of view of reliability, long life, expediency of using a compressor of maximum possible Isentropic efficiency, the new unit has all these factors in its favour.

(a) The compressors of the unit with the new scheme operate at constant speed and flow parameters as well as efficiency at all loads i.e. they operate with a single point characteristics. Designing such a compressor with the maximum Isentropic efficiency would therefore be a worthwhile aim to achieve. The designer would use all the available technique both aerodynamically and mechanically in obtaining this result. No compromise nor sacrifice of one consideration for the sake of another is necessary since there is no variation in running conditions in this case.

(b) In the case of variable speed compressor, stage-matching occurs only at the design point while stage mismatching is bound to occur at other points in the operating range. The compressor of the unit with the new scheme would not at any time suffer from this

stage-mismatching. It would therefore be expected to have a much longer life and be more reliable.

(c) With the present scheme, the compressors although mounted on the same shaft would not suffer from component-mismatching *i.e.* tendency of one to surge while the other to choke, since they run at constant speed at all loads.

4. It is well known that the layout of the system of a gas-turbine plant can have a very important effect on the plant's overall performance. In particular, ducting when excessive as is sometimes necessary with extended layout causes a loss in the developed power of as much as 20 % .

In the present scheme, the minimum of ducting is necessary since the layout is obviously simple.

5. The main components of a gas-turbine unit form a pneumatic system dealing with a single fluid. It is therefore a quite suitable machine for applying standardisation of components. This is particularly so in the case of the compressors of the present scheme, since they operate on a one point characteristics while the range of actual flow could be varied widely. A compressor of a particular size and specifications would thus be suitable in a wider range and variety of gas turbine applications. Moreover, the incorporation of an air turbine in the unit in the way proposed here would ensure a greater flexibility in choosing a standard compressor.

Quoting Hayne Constant⁽¹⁶⁾, "If such a process of standardisation proves to be practicable, we may see a situation develop in which specialist firms concentrate on the production of a limited range of components while assembly firms assemble them into complete gas-turbine systems. The virtual disappearance of the made-to-measure gas-turbine would certainly result in a considerable decrease in costs".

6. It is seen from the curve Fig. 10 of adiabatic efficiency against throughput for the air turbine that the adiabatic efficiency is reasonably high at high throughputs and low at low throughputs. This is a result of the method of varying the throughput. This method

consists of rotating the guide blades (which act as admission nozzles), Figs. (7, 8 and 9) such that the actual passages vary in width while the inlet angles as well, change.

At very small throughputs, narrow passages result in high losses and hence the steep drop in the air-turbine efficiency.

However, it is to be noted that the low adiabatic efficiency of an air turbine incorporated in a gas-turbine unit as proposed in the present scheme, will have little effect on the overall thermal efficiency of the unit: at small part loads the mass flow in the air-turbine is relatively a small part of the total flow. Hence, the air-turbine power at that low adiabatic efficiency would be too small to have any appreciable effect on the power developed by the unit.

The new application of the air-turbine suggested here should draw the attention of designers to applying the most useful technique for producing a suitable machine having as flat an adiabatic efficiency curve as possible.

APPENDIX

To find the Optimum Pressure Ratio for the L. P. Compressor in the proposed Scheme presented here for the power control of constant speed Gas-Turbines.

- Let R = Overall Pressure Ratio of Compression
 R_1 = Pressure Ratio of L. P. Compressor
 R_2 = Pressure Ratio of H. P. Compressor
 x = Air flow in H. P. compressor as a fraction of
Air flow in L. P. Compressor
 $\therefore (1 - x)$ = Air flow in Air-Turbine as a fraction of Air
flow in L. P. Compressor
 y = Ratio of pressure at outlet to pressure at inlet
of Intercooler
 $(1 - y)$ = Pressure loss in Intercooler as a fraction of
Pressure at inlet to intercooler

Pressure after throttling = Pressure at inlet to Intercooler
 = $x \times$ Pressure at outlet of L.P.
 Compressor

$\therefore$ Pressure at outlet of Intercooler = $x \times y \times$ Pressure at
 outlet of L.P. Compressor

$$\therefore R_2 = \frac{R}{xy R_1}$$

$T_1 + t$ = Temp. at inlet to L. P. compressor

T_1 = Temp. at inlet to L. P. compressor

η_c = Isentropic Efficiency of L.P and H.P. compressors

$\eta_{A.T.}$ = Isentropic Efficiency of Air-Turbine

Z = Rise in Air Temperature in the Auxiliary Heat
 Exchanger.

$\therefore$ Net Compression work per lb. of Air flow in L. P. compressor

$$W = C_p T_1 (R^{\frac{\gamma-1}{\gamma}} - 1) \times 1/\eta_c + x C_p (T_1 + t) \left[\left(\frac{R}{xy R_1} \right)^{\frac{\gamma-1}{\gamma}} - 1 \right] \\ \times 1/\eta_c - (1-x) C_p (T_1 R_1 + Z) \times [1 - (1/R_1)^{\frac{\gamma-1}{\gamma}}] \times \eta_{A.T.}$$

Differentiating Partially with respect to R_1 ,

$$\frac{\partial W}{\partial R_1} = C_p T_1 \times \frac{1}{\eta_c} \times \frac{\gamma-1}{\gamma} R_1^{-1/\gamma} + x C_p (T_1 + t) \times \frac{1}{\eta_c} R_1^{\frac{\gamma-1}{\gamma}} \\ \times (xy)^{-\frac{\gamma-1}{\gamma}} \times \frac{1-\gamma}{\gamma} R_1^{\frac{1-2\gamma}{\gamma}} - (1-x) C_p T_1 \times \eta_{A.T.} \\ \times \frac{\gamma-1}{\gamma} R_1^{-1/\gamma} + (1-x) C_p Z \times \frac{1-\gamma}{\gamma} R_1^{\frac{1-2\gamma}{\gamma}} \times \eta_{A.T.}$$

$$\text{For Min. } W, \frac{\partial W}{\partial R_1} = 0$$

$$\begin{aligned} \therefore 1/\eta_c \times \frac{\gamma-1}{\gamma} R_1^{-1/\gamma} + x \frac{T_1+t}{T_1} \times 1/\eta_c R^{\frac{\gamma-1}{\gamma}} (xy)^{-\frac{\gamma-1}{\gamma}} \\ \times \frac{1-\gamma}{\gamma} R_1^{\frac{1-2\gamma}{\gamma}} - (1-x) \times \eta_{A.T.} \times \frac{\gamma-1}{\gamma} \\ R_1^{-1/\gamma} + \eta_{A.T.} (1-x) \times \frac{Z}{T_1} \times \frac{1-\gamma}{\gamma} R_1^{\frac{1-2\gamma}{\gamma}} = 0 \end{aligned}$$

We get $(R_1^2)^{\frac{\gamma-1}{\gamma}}$

$$\begin{aligned} & x \frac{T_1+t}{T_1} (xy)^{\frac{1-\gamma}{\gamma}} R^{\frac{\gamma-1}{\gamma}} + \eta_{A.T.} \eta_c (1-x) \frac{Z}{T_1} \\ & = \frac{1 - (1-x) \eta_{A.T.} \eta_c}{1 - (1-x) \eta_{A.T.} \eta_c} \end{aligned}$$

$$\therefore R_1 = \sqrt{\left[\frac{x^{1/\gamma} (T_1+t) (R/\gamma)^{\frac{\gamma-1}{\gamma}} + \eta_{A.T.} \eta_c (1-x) Z}{T_1 [1 - (1-x) \eta_{A.T.} \eta_c]} \right]^{\gamma/\gamma-1}}$$

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“FORCE-EXTENSION CHARACTERISTIC OF AN ELASTIC SYSTEM DYNAMICALLY LOADED BY AN IMPULSE.”

BY

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INTRODUCTION

In dealing with the analysis of the effect of impulsive forces in machine design, different assumptions are often made to simplify the problem. The formulae obtained are usually tailored to fit certain dynamic conditions.

In this work, the fundamental approach to the problem will be followed and verified with tests. This approach should furnish a general and satisfactory solution to the problem of mechanical systems under impulse.

A single degree of freedom system will be considered in this work.

THE EXPERIMENTAL EQUIPMENT

The elastic system used in this work comprises a rectangular table $24" \times 12" \times 3/4"$ supported by four leaf springs fixed to the floor. The weight of the table, added to it one third of that of the four springs, is 76 pounds. A metallic hammer suspended from the ceiling by four wires in a pendulum fashion of 138.8" radius, was used to strike the elastic table. Since the radius of the pendulum was large and its angular displacement was small, the hammer could be assumed as hitting the table horizontally in a straight line path.

Two hammers of different weights were used, namely, 2.6 lb. and 5.2 lb. The hammers are made of aluminium alloy cylinders, $2\frac{1}{4}$ " diameter, fitted with a spherical bronze cap, $2\frac{1}{4}$ " radius.

A potentiometer type pick-up unit was used to record the displacement-time curves for the table, when struck by the hammer, on a brush recorder. This recorder was first calibrated by displacing the table one half inch either way from the equilibrium position and recording this one inch amplitude on the recorder. The speed of the recording paper was 124 mm/sec.

An arc, graduated into angular degrees and carried by a stand, was placed close to the swinging hammer to measure its angle of release and its angular rebound. The arrangement is shown in Figure 1.

The velocities of the hammer immediately before and after impact were determined from the potential energies in the initial and the rebound positions.

$$v = \sqrt{2gh} = \sqrt{2gl(1 - \cos \theta)} \quad . \quad . \quad . \quad (1)$$

where v = velocity of impact or rebound of hammer.

l = length of radius of pendulum.

θ = angle of release or rebound with the vertical.

The velocity imparted to the table is calculated from the tangent to the displacement-time relationship recorded on the recorder by the potentiometer pick-up.

The duration of contact of the two colliding bodies, namely, the hammer and the table, was estimated from a formula suggested by Professor W.T. Thomson, and which was confirmed by unpublished tests carried out by recording the continuity of an electric circuit when a swinging ball hit a vertical plate.

$$t = 3.69 \times 10^{-4} R.v^{-0.2} \quad . \quad . \quad . \quad (2)$$

where t = time of contact in seconds.

R = radius of striking sphere in inches.

and v = velocity of impact in inches/sec.

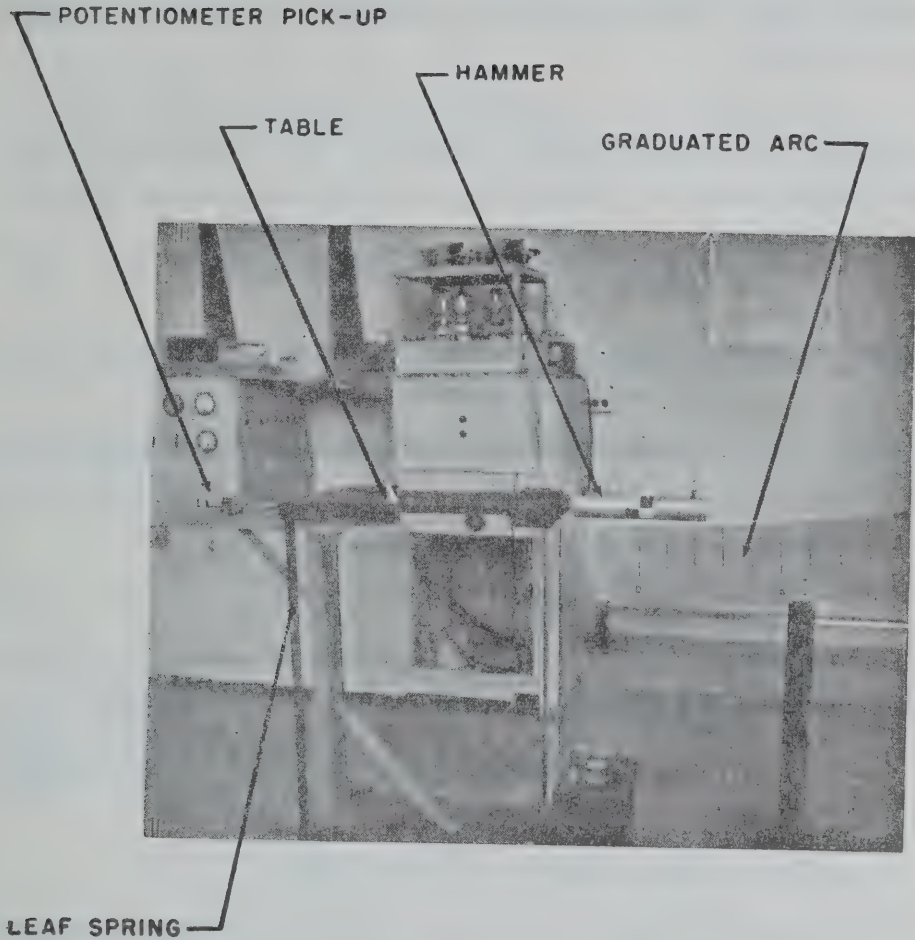


FIG. 1.—Experimental Equipment

THEORETICAL BACKGROUND

In impact, characterized by the collision of two bodies, usually the duration of contact is short and high transient stresses are locally introduced. The analysis of the impact involves certain fundamental concepts of mechanics, namely :

1. Conservation of mass.
2. Conservation of momentum.
3. Conservation of energy.

For the two body system, the satisfaction of the first of these is obvious which leaves the proper application of the last two concepts to be fulfilled.

The conservation of momentum during impact follows from the fact that, by virtue of Newton's third law, the impulse for the two body system is zero. The momentum relationship can be expressed by the equation.

$$m v + M V = m v' + M V'.$$

where $v, v' =$ velocity of hammer mass "m" immediately before and after impact.

$V, V' =$ velocity of table mass "M", immediately before and after impact.

In the present work $V = 0$ and hence.

$$m v = m v' + M V'. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In applying the concept of conservation of energy, it is noted that the initial kinetic energy of the hammer is transformed into:-

1. The kinetic energy of the hammer and the table after impact,
2. The work of plastic deformation of the bodies, and
3. The vibration energy of colliding bodies.

The last two forms of energy are eventually dissipated and can be classified under loss of energy. Thus the energy balance can be expressed by the equation.

$$\text{Loss of energy} = \frac{1}{2} m v^2 - \left(\frac{1}{2} m v'^2 + \frac{1}{2} M V'^2 \right) \quad . \quad (4)$$

On the other hand, the coefficient of restitution is given by

$$e = \frac{V' - v'}{v} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and by the substitution of Eqs. (3) and (5) in Eq. (4) we get

$$\text{Loss of energy} = \frac{1}{2} (1 - e^2) \frac{m M}{m + M} \cdot v^2 \quad . \quad . \quad . \quad . \quad (6)$$

The motion of the table can be determined by considering the free vibration of M with the initial conditions, at $t=0$, $x = 0$ and $\dot{x} = V'$.

Starting with the general solution,

$$x = X \cdot e^{-\zeta \omega_n t} \cdot \sin(\sqrt{1 - \zeta^2} \omega_n t - \phi) \quad (7)$$

where $\omega_n = \sqrt{\frac{K_a}{M}} = \text{natural frequency rad/sec.}$

$$\zeta = \frac{c}{c_{cr}} \simeq \frac{1}{2\pi} \ln \frac{x_n}{x_{n+1}} = \text{damping factor,}$$

the substitution of the boundary conditions, mentioned before, results in the solution

$$x = \frac{V'}{\omega_n \sqrt{1 - \zeta^2}} \cdot e^{-\zeta \omega_n t} \cdot \sin \sqrt{1 - \zeta^2} \omega_n t \quad (8)$$

If we consider the table M by itself, the impulse I imparted to it by the hammer is equal to the change in its momentum MV' . Thus the equation (8) can alternately be written as:

$$x = \frac{I}{M \omega_n \sqrt{1 - \zeta^2}} \cdot e^{-\zeta \omega_n t} \cdot \sin \sqrt{1 - \zeta^2} \omega_n t$$

where $I = \int F_i dr = MV'$

At $\omega_n t = \frac{\pi}{2} = \frac{\tau}{4}$, $\sin \sqrt{1 - \zeta^2} \omega_n t = 1$

and we get

$$I = e^{\frac{\pi}{2} \zeta} \sqrt{M K_s (1 - \zeta^2)} \cdot x_{\tau/4} \quad (9)$$

and similarly

$$F_i = \frac{I}{t_s} = e^{\frac{\pi}{2} \zeta} \sqrt{M K_s (1 - \zeta^2)} \cdot \frac{x_{\tau/4}}{t_s} \quad (10)$$

where F_i is the average impulsive force in lb. and t_0 is the time of force application in seconds.

Equations (9) and (10) give the relationships correlating the displacement $x_{\tau/4}$ to the impulse I imparted to the system and to the impulsive force F_i respectively.

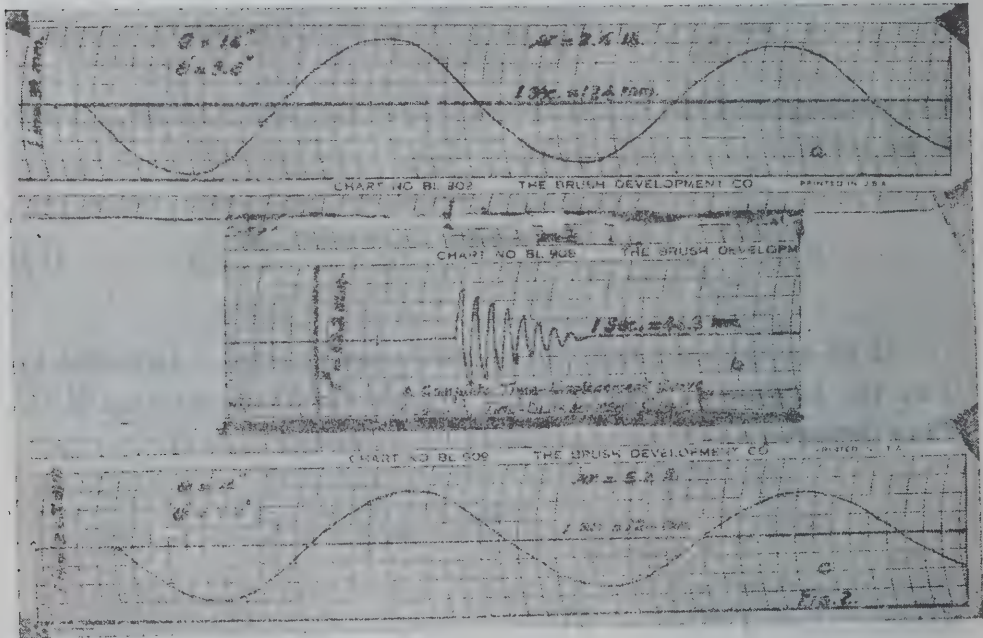


FIG. 2

EXPERIMENTAL RESULTS

Tests were carried out using two hammers of different weights, namely, 2.6 and 5.2 pounds. The displacement-time curves were obtained in each case for various release angles. Curves a and c given in figure 2 show examples for testing with 2.6 and 5.2 lb. respectively while the curve in b gives a sample of a complete curve recorded at a slower speed. The computed data is tabulated in Table I.

TABLE I

	w	θ	θ'	v	v'	V	V'	e	L
No.	Weight of Hammer lb	Release angle θ degrees	Rebound angle θ' degrees	Hammer vel. before Impact in/sec	Hammer vel. after impact in/sec	Table vel. before impact in/sec	Table vel. after impact in/sec	Coeff't of res-titution	Energy loss lb. in.
1	5.2	5	3	20.2	-12.2	0	2.55	0.732	1.185
2	5.2	7	4.2	28.3	-17.6	0	3.30	0.739	2.28
3	5.2	9	5.8	36.4	-23.4	0	3.73	0.746	3.53
4	5.2	10	6.9	40.4	-26.2	0	4.45	0.758	4.37
5	5.2	11	7.2	44.3	-29.0	0	5.05	0.77	5.01
6	2.6	14	8.3	56.5	-33.5	0	3.02	0.646	6.07
7	2.6	16	9.8	64.2	-39.5	0	3.54	0.669	7.5
8	2.5	18	10.8	72.6	-43.5	0	3.94	0.656	9.78
9	2.6	20	11.9	80.2	-48.0	0	4.35	0.651	12.15

TABLE I (Continued)

	ζ	I	w	x	v	t_0	F_1
No.	Damping factor	Impulse lb/sec.	lb	in	in/sec.	μ sec.	lb.
1	0.029	0.5	5.2	0.329	20.2	443.8	1130
2	0.029	0.649	5.2	0.426	28.3	415.1	1560
3	0.029	0.732	5.2	0.482	36.4	394.8	1860
4	0.029	0.871	5.2	0.575	40.4	386.6	2260
5	0.029	0.99	5.2	0.650	44.3	379.5	2600
6	0.027	0.595	2.6	0.390	56.5	361.5	1650
7	0.026	0.696	2.6	0.457	64.2	352.4	1970
8	0.026	0.776	2.6	0.510	72.6	343.8	2250
9	0.025	0.861	2.6	0.565	80.2	337.0	2550

The stiffness K_s of the system was calibrated by statically loading the table horizontally and measuring the corresponding extension. The results are shown in Figure 3 which gave $K_s = 11.4$ lb/in. In the meantime when K_s was computed from the displacement-time curves, it was found to be 10.8 lb/in. and this discrepancy in its value is attributed mainly to damping effect.

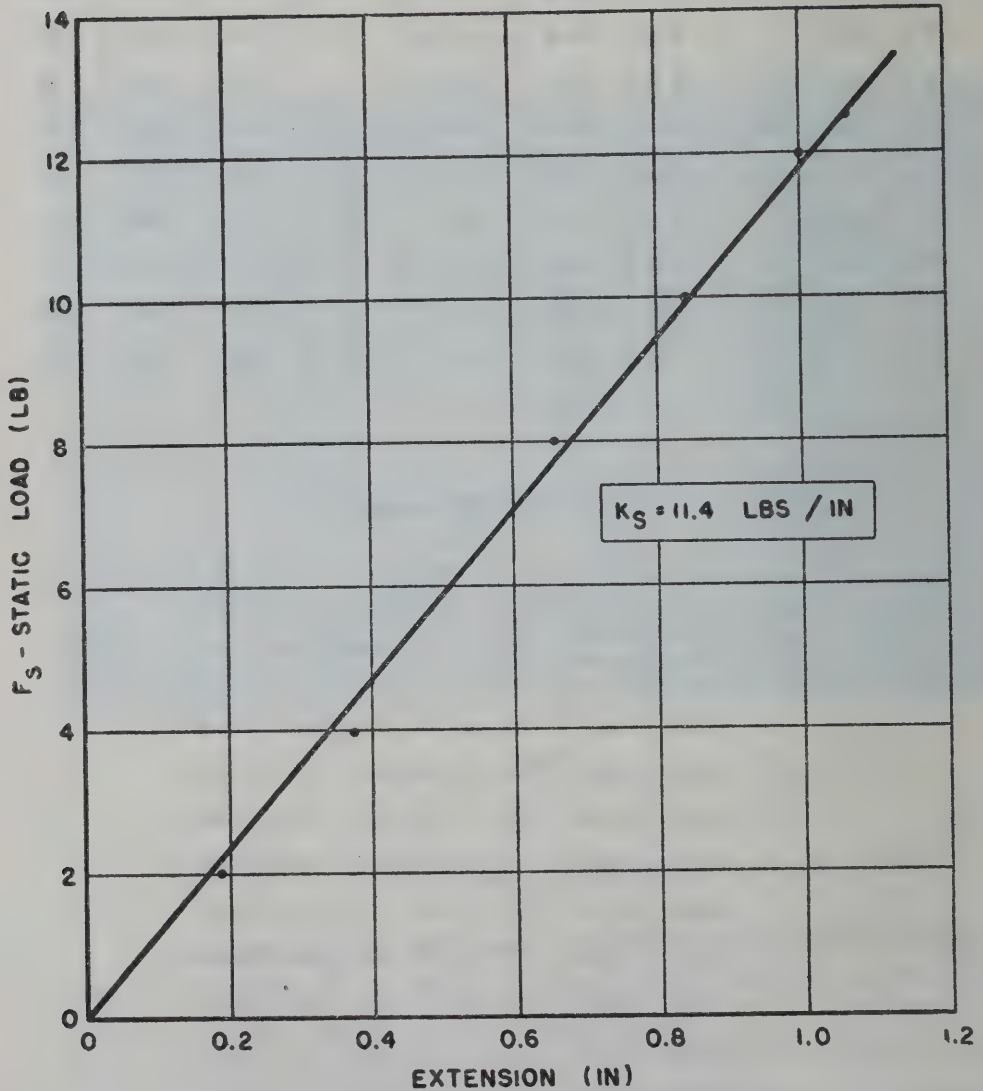


FIG. 3.— Calibration of K_s , the Static Spring Constant of the System

The experimental and theoretical results, establishing a straight line relationship between the impulse and the displacement of table at quarter the period of vibration, both agreed very closely as shown in Figure 4.

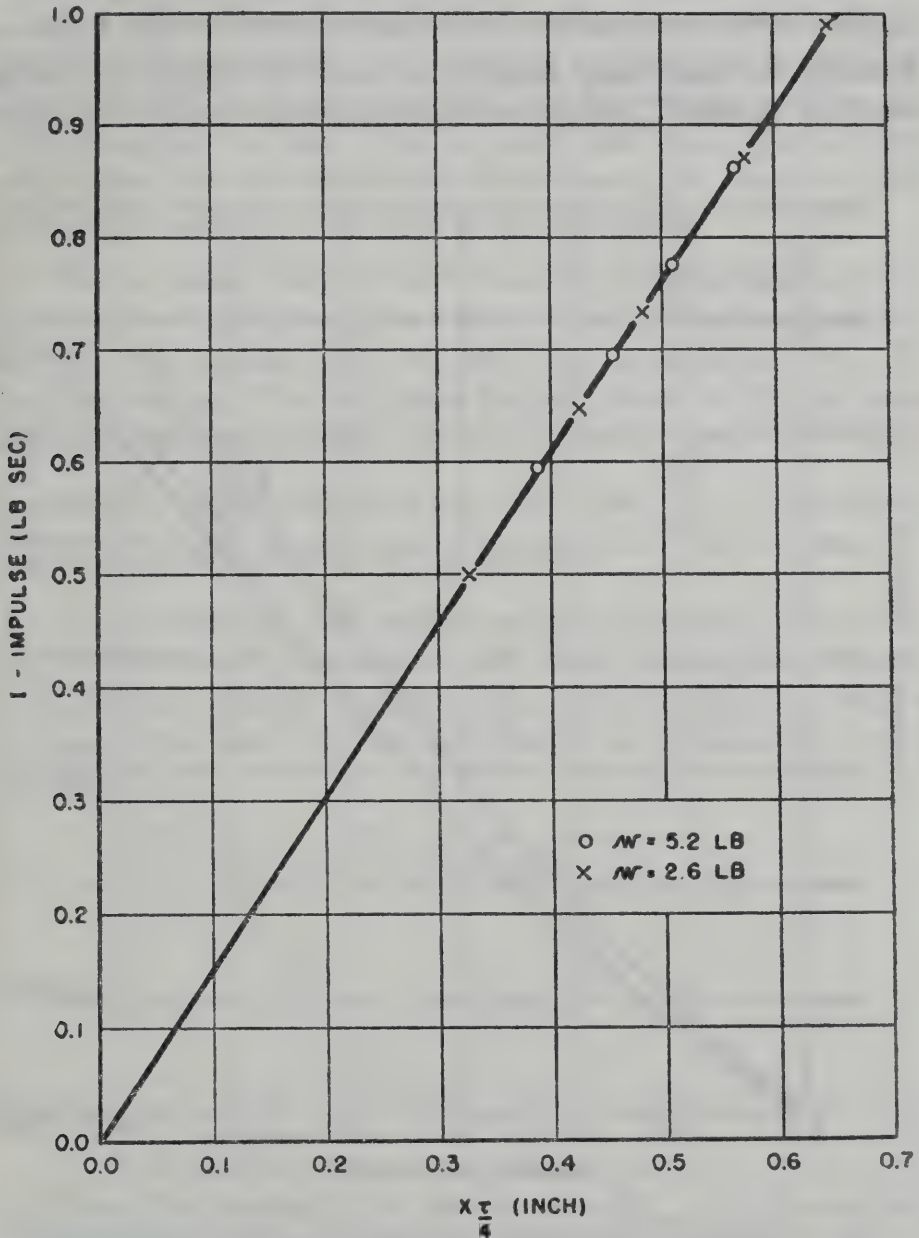


FIG. 4.—Relation between Impulse and Extension

The times of contact were calculated using equation (2) and the average values of the impulsive force were obtained by dividing the impulse I by the time of contact t_0 .

When the impulsive force F_i was plotted against the displacement $x_{\tau/4}$ two curves were obtained, one for each hammer weight, Figure 5. Although the relationship seems to be almost a straight line one at very high values of F_i , yet at lower values this is not true since the graph

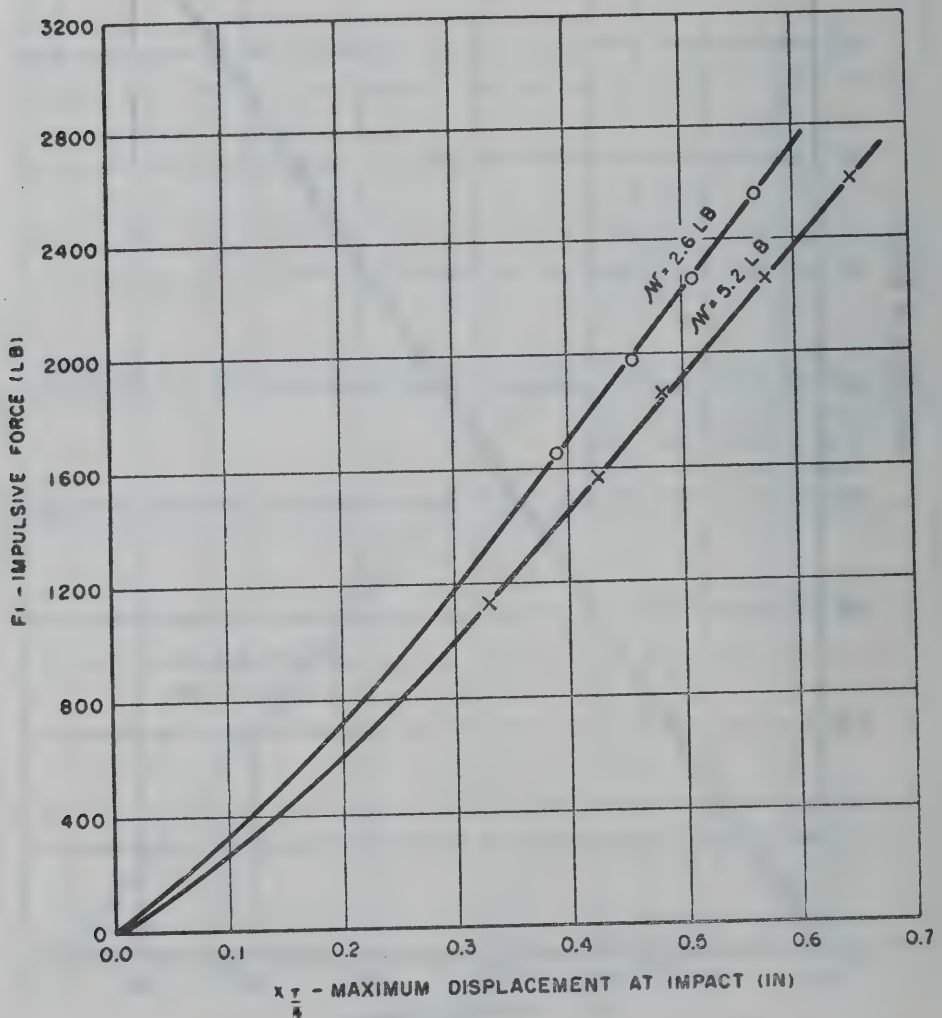


FIG. 5.—Change of Maximum Displacement $x_{\tau/4}$ with Impulsive Force

should pass by the origin. However, it was difficult to obtain readings in the lower range of the impulse force with the present equipment due to the lack of experimental accuracy in this range of small angles of release and rebound.

It is evident from Figure 5, that for the same extension a bigger impulsive force was needed when a hammer of lighter weight was used. This is justified by the fact that with the lighter weight hammer, the time of contact was less. This is again clear from equation (10) which gives, for the same value of extension, an impulsive force inversely proportional to the time of its application.

For the same values of extension, the corresponding values of F_i and I were taken from figures 5 and 4 respectively and consequently the times of impulse were computed and tabulated in Table II. Applying a term K_i for impulsive loading, similar to K_s the static stiffness of the elastic system, a curve is shown in figure 6 correlating K_i with the dimensionless value t_o/τ where $K_i = \frac{F_i}{x_{r/4}}$, t_o is the time of impulse and τ is the period of free vibration.

It is evident that the system has not one single value of the impulsive stiffness K_i as is the case with static loading, but the value of K_i depends mainly upon the ratio t_o/τ and it increases with a decrease in the later. This is illustrated by the equation (10) but in the form.

$$\frac{F_i}{x_{r/4}} = K_i = e^{\frac{\pi}{2} \zeta} \sqrt{M K_s (1 - \zeta^2)} \cdot \frac{1}{t_o}$$

Putting $\omega_n = \sqrt{\frac{K_s}{M}}$ and $\tau = 2\pi \sqrt{\frac{M}{K_s}}$

$$\text{therefore } K_i = \frac{1}{2\pi} \cdot e^{\frac{\pi}{2} \zeta} \sqrt{1 - \zeta^2} \cdot K_s \cdot \frac{1}{t_o/\tau} \quad (11)$$

From this equation, it is obvious that as $t_o/\tau \rightarrow 0$, $K_i \rightarrow \infty$ and conversely as $t_o/\tau \rightarrow \infty$, $K_i \rightarrow 0$.

TABLE II

x in.	F_i lb	I lb. sec.	t_o μ sec	$K_i = \frac{F_i}{x}$ lb/in.	$\frac{t_o}{\tau}$ $\times 10^{-6}$	$\frac{K_i}{K_s}$
0.05	110	0.075	681	2200	800	196.5
0.05	150	0.075	500	3000	589	268
0.10	250	0.150	600	2500	705	223
0.10	320	0.150	468	3200	550	286
0.15	415	0.228	550	2760	648	247
0.15	510	0.223	447	3400	525	304
0.20	590	0.304	514	2950	604	264
0.20	720	0.304	422	3600	495	321
0.25	780	0.380	486	3120	572	279
0.25	940	0.380	404	3760	476	336
0.30	1000	0.459	456	3333	537	297
0.30	1180	0.459	386	3933	454	351
0.35	1215	0.532	433	3470	510	311
0.35	1440	0.532	370	4110	435	367
0.40	1440	0.610	423	3600	497	321
0.40	1700	0.610	358	4250	421	380
0.45	1665	0.685	411	3700	484	330
0.45	1960	0.685	349	4350	410	388
0.50	1900	0.760	400	3800	471	339
0.50	2200	0.760	345	4400	406	392
0.55	2120	0.837	394	3860	463	344
0.55	2460	0.837	340	4470	400	400
0.60	2360	0.912	386	3920	453	350
0.60	2720	0.912	335	4530	394	404

$\tau = 0.85$ sec.

In the meantime when the time of impact t_0 is equal to the period of free vibration of the system,

$$K_i = \frac{1}{2\pi} e^{\frac{\pi}{2}\zeta} \cdot \sqrt{1-\zeta^2} \cdot K_s \simeq \frac{1 + \frac{\pi}{2}\zeta}{2\pi} \cdot K_s \quad (12)$$

In a similar way it can be shown that the impulsive stiffness of an elastic system will be equal to its static stiffness K_s when

$$t_0 \simeq \frac{1 + \frac{\pi}{2}\zeta}{2\pi} \cdot \tau \quad \dots \dots \dots (13)$$

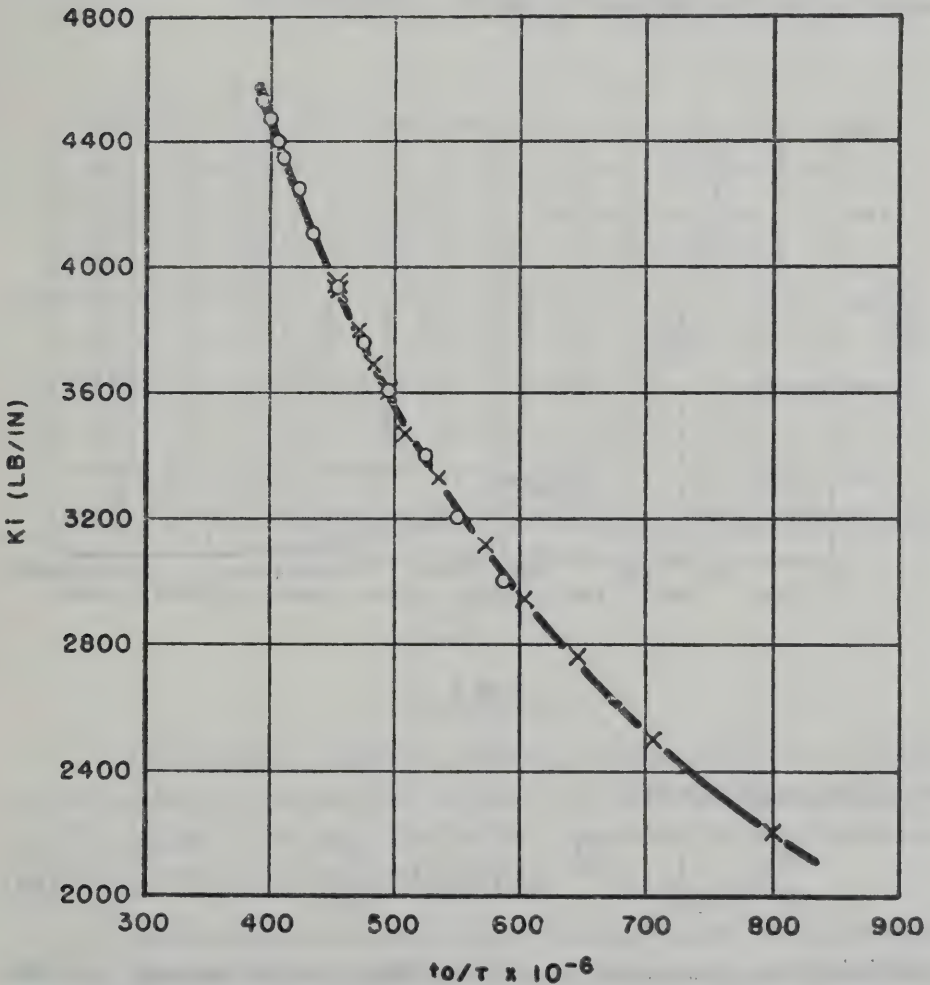


FIG. 6.

However, in order to be able to give a full picture of the behavior of K_i relative to changes in t_o/τ , though limited to the present value of damping factor ζ and from a theoretical point of view, the values of K_i and t_o/τ were calculated and plotted on a semi-logarithmic basis in Figure 7, passing by the extreme points of $t_o/\tau = 1$ and $K_i = K_s$.

In order to find out if the damping factor ζ was affected by the time of impulse or not, the same curve in Figure 7 was drawn on a full logarithmic basis and the result was a straight line which proved that the value of ζ was not affected by the time of impulse, since the relation between K_i and t_o/τ is hyperbolic.

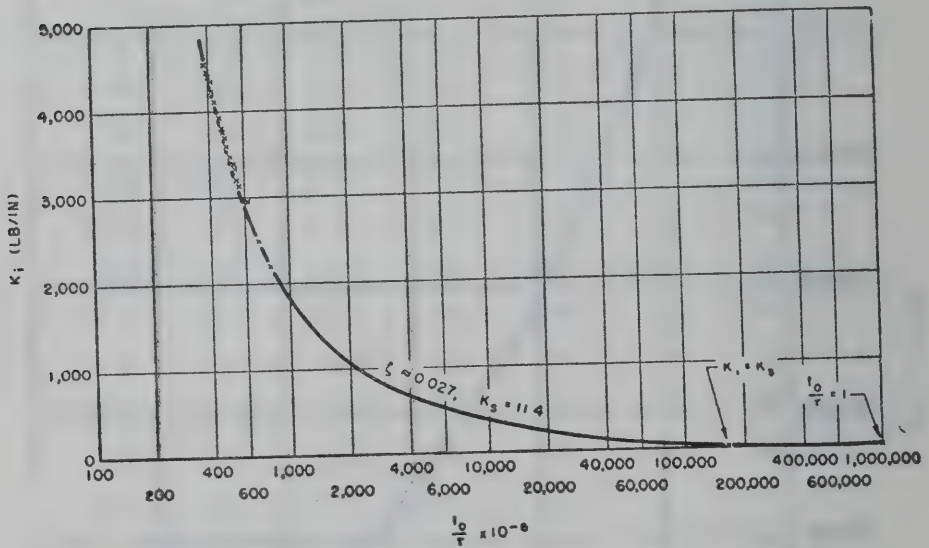


FIG. 7

From equation (11)

$$\frac{K_i}{K_s} = \frac{1}{2\pi} \cdot e^{\frac{\pi}{2}\zeta} \sqrt{1-\zeta^2} \cdot \frac{1}{t_o/\tau} \quad (14)$$

which implies that a more generalized family of curves can be drawn correlating two dimensionless values, namely, K_i/K_s and t_o/τ for

various values of ζ . One curve of this family is shown in Figure 8 which is obtained in the present work for an elastic system having a damping factor equal to about 0.027. The position of the curves, corresponding to other values of ζ , relative to the curve, shown in Figure 8, will mainly depend upon the value of

$$S = \frac{1}{2\pi} \cdot e^{\frac{\pi}{2}\zeta} \cdot \sqrt{1 - \zeta^2},$$

which in turn depends upon the damping factor ζ .

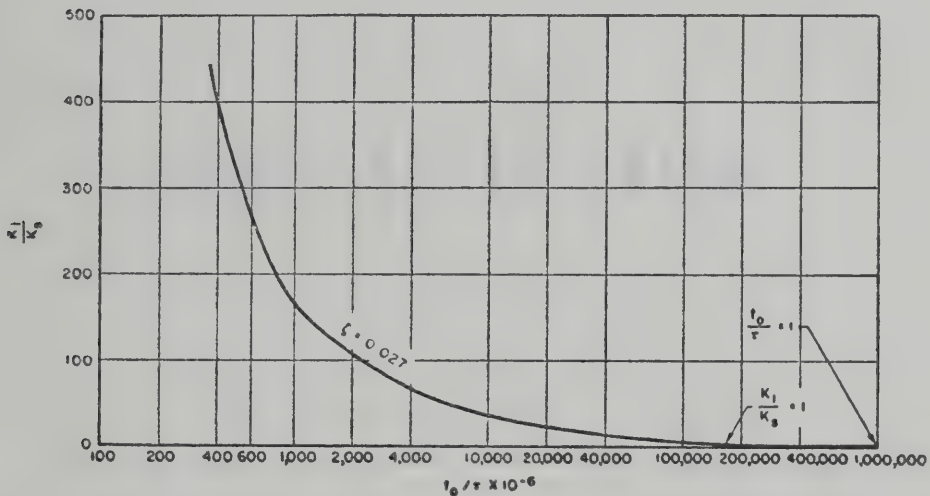


FIG. 8

CONCLUSIONS

1. The dynamic extension resulting on the spring for a certain elastic system is a function of impulse only, namely, it is not affected by the change in the magnitude of the impulsive force or its duration as long as their product stays constant.

2. The extension obtained due to the application of an impulsive force to an elastic system does not bear a constant proportionality to the applied force.

3. The impulsive force per unit extension, namely, the impulsive stiffness K_i , depends upon the time of application of this force, the damping factor and the static stiffness K_s of the elastic system.

4. The relation between the two dimensionless factors K_i/K_s is a hyperbolic one which is defined mainly by the damping factor of the system. This relationship can be represented by a family of hyperbolas each corresponding to a certain damping factor ζ .

AERO. ENGINEERING

DESIGN OF BLADE CASCADES WITH MINIMUM MAXIMUM VELOCITY

BY

M. I. I. RASHED., Dr. Sc. tech.

INTRODUCTION

The trial to solve the problem of cascade design, using the methods of Hydrodynamics goes back to a long time ago when the engineers were faced with the construction of special rotary machines (1). The unreliability of the triangles of velocities to fix the inlet and outlet directions of the blades is well known (2). It had been a current practice (3) to have the angles taken from the triangle of velocity corrected in an impirical manner to get the blade angles.

The use of the lift and drag characteristics of the blade cascades to the design of axial turbomachines was discussed by Ed. Amstutz (4). The application of the results obtained from plane cascade testing is based on the assumption that the flow in an axial turbomachine takes place on co-axial cylindrical surfaces. The development of these surfaces produces the flow pattern through a plane cascade.

The action of a plane cascade is to deflect the flow and this is achieved by loading the blades of the cascade in a certain manner which is specified by the method or the theory applied to design the cascade. The theories are classified as follows :

1. Theories which give the characteristics of a pre-designed cascade arrangement.
2. Theories which give the design of the cascade characterised by certain pre-specified requirements.

The same theories investigating cascades can be differently classified :

1. Theories depending on the method of conformal representation (4-8).

2. Theories in which the whole or parts of the blades are replaced by either point or continuously distributed hydrodynamical singularities^(9,10). Belonging to this category is the Ackeret cascade theory⁽¹⁰⁾.

THE ACKERET CASCADE THEORY

Before presenting the theory itself, the following principle is to be stated:

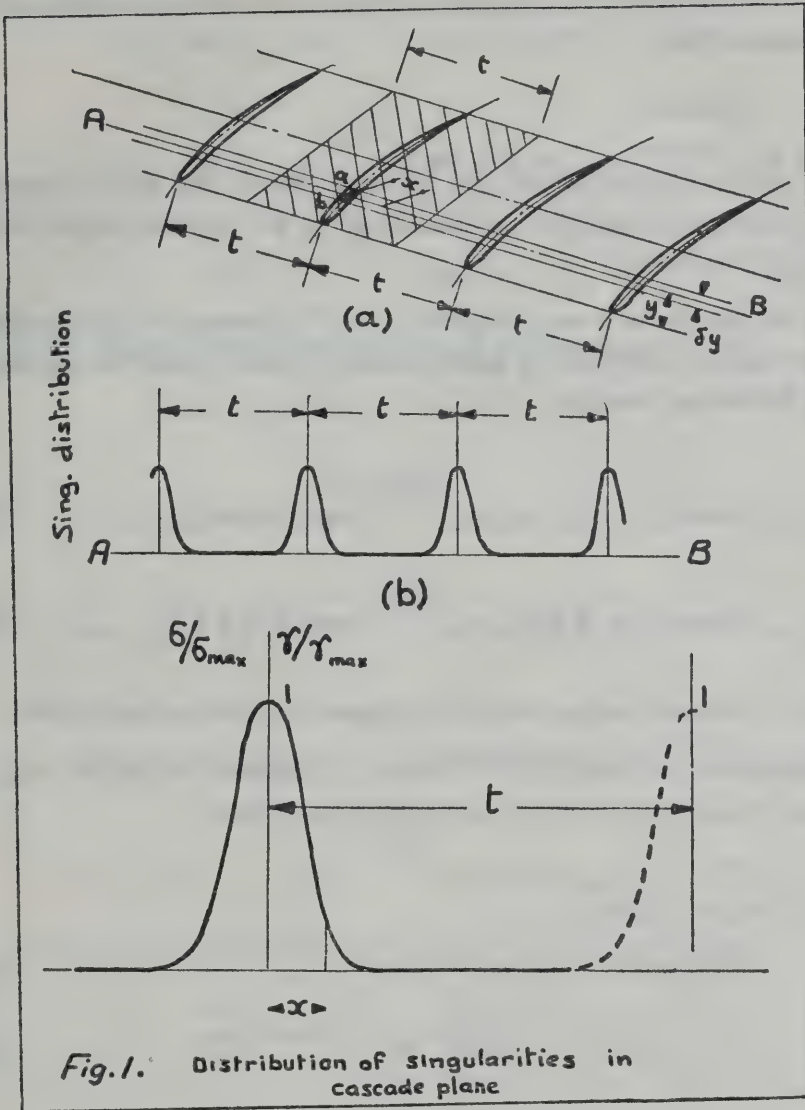
“The flow round any body and hence round any airfoil section or round a system of bodies can be obtained by replacing the bodies by hydrodynamical singularities, i.e. sources, sinks and vortices^(12,13)”.

Hydrodynamical singularities give rise to mathematical difficulties. To get rid of these difficulties, Prof. Ackeret suggested⁽¹¹⁾ the distribution of these singularities all over the area so that, the source, the sink or the vorticity per unit area at any point would be finite. This distribution of singularities permits a quicker computation of the resultant velocity field.

Distribution of singularities:

Consider the elementary strip A B in the cascade plane as shown in figure 1a. Along the strip A B the distribution of the singularities is as shown in figure 1b. The effect of the singularities in this strip on the flow pattern can be studied by analysing the singularity distribution in a Fourier series. The total effect is the sum of effects of the mean value and the harmonics.

The circulation around one blade determines the vorticity in the area between the center lines of two adjacent blades, i.e. in the shaded area shown in figure 1a. The sources or sinks in the cascade plane are such that if the sources or sinks in the shaded part of the elementary strip A B in Figure 1a are concentrated on the elementary length ab of the center line of the corresponding blade, then the effect of the source and sink distribution thus achieved gives rise to the blade section.



Distribution curve :

The distribution of the sources, sinks or vortices is to be as follows.

$$\gamma/\gamma_{\max} = e^{-\xi_t^2/a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1a)$$

$$\sigma/\sigma_{\max} = e^{-\xi_t^2/a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1b)$$

where σ and γ denote the source (or sink) and the circulation at the blade center line.

$$\xi_t = x/t$$

x = the distance from the point at which the source strength or the circulation strength is considered to the nearest center line as shown in Figure 1a.

This distribution was adopted by W.T. Sawyer (14) in his thesis. γ and σ can be analysed to mean values γ_o and σ_o and to harmonics in the following manner.

$$\gamma = \gamma_o + \sum 2 \gamma_o \cdot e^{-\left(\frac{n}{5}\right)^2} \cdot \cos (2 \pi n \xi_t) \quad (2a)$$

$$\sigma = \sigma_o + \sum 2 \sigma_o \cdot e^{-\left(\frac{n}{5}\right)^2} \cdot \cos (2 \pi n \xi_t) \quad (2b)$$

Effect of the mean values and the harmonics. on the velocity field :

Consider the strip AB of width δy shown in Figure 1a, the effect of the singularities in this strip is as follows :

Effect of mean values :

$$\delta v_{\gamma_o} = 0, \delta u_{\gamma_o} = \gamma_o \cdot \delta y \quad (3a)$$

$$\delta u_{\sigma_o} = 0, \delta v_{\sigma_o} = \sigma_o \cdot \delta y \quad (3b)$$

Effect of harmonics :

At a point (x, y) lying in the upper side, with

$$\xi_t = x/t \text{ and } \eta_t = y/t$$

$$\text{we get } \delta u_{\gamma_n} = \gamma_o \cdot e^{-\left(\frac{n}{5}\right)^2} \cdot \cos (2 \pi n \xi_t) \cdot e^{-2 \pi n \eta_t} \cdot \delta y \quad (4a)$$

$$\delta v_{\gamma_n} = \gamma_o \cdot e^{-\left(\frac{n}{5}\right)^2} \cdot \sin(2\pi n \xi_t) \cdot e^{-2\pi n \eta_t} \cdot \delta y \quad (4b)$$

$$\delta v_{\sigma_o} = \delta_o \cdot e^{-\left(\frac{n}{5}\right)^2} \cdot \cos(2\pi n \xi_t) \cdot e^{-2\pi n \eta_t} \cdot \delta y \quad (4c)$$

$$\delta u_{\sigma_n} = \delta_o \cdot e^{-\left(\frac{n}{5}\right)^2} \cdot \sin(2\pi n \xi_t) \cdot e^{-2\pi n \eta_t} \cdot \delta y \quad (4d)$$

For a point in the lower side ($y = \text{negative}$) and with

$$\xi_t = x/t \text{ and } \eta_t = -y/t$$

$$\delta u_{\gamma_n} = -\gamma_o \cdot e^{-\left(\frac{n}{5}\right)^2} \cdot \cos(2\pi n \xi_t) \cdot e^{-2\pi n \eta_t} \cdot \delta y \quad (5a)$$

$$\delta v_{\gamma_n} = -\gamma_o \cdot e^{-\left(\frac{n}{5}\right)^2} \cdot \sin(2\pi n \xi_t) \cdot e^{-2\pi n \eta_t} \cdot \delta y \quad (5b)$$

$$\delta v_{\sigma_n} = -\delta_o \cdot e^{-\left(\frac{n}{5}\right)^2} \cdot \cos(2\pi n \xi_t) \cdot e^{-2\pi n \eta_t} \cdot \delta y \quad (5c)$$

$$\delta u_{\sigma_n} = \delta_o \cdot e^{-\left(\frac{n}{5}\right)^2} \cdot \sin(2\pi n \xi_t) \cdot e^{-2\pi n \eta_t} \cdot \delta y \quad (5d)$$

Comparing equations 4 and 5 it can be easily proved that

$$\frac{1}{\gamma_o} \cdot \frac{d u_{\gamma_n}}{d y} = \frac{1}{\sigma_o} \cdot \frac{d v_{\sigma_n}}{d y} \quad (6a)$$

$$\frac{1}{\gamma_o} \cdot \frac{d v_{\gamma_n}}{d y} = -\frac{1}{\sigma_o} \cdot \frac{d u_{\sigma_n}}{d y} \quad (6b)$$

Adding all the harmonics to get the values for

$$\frac{1}{\gamma_o} \cdot \frac{d u_{\gamma H}}{d y} \text{ and } \frac{1}{\gamma_o} \cdot \frac{d v_{\gamma H}}{d y}$$

The values of which are drawn for positive values of ξ_t and η_t in plates I and II.

Velocity at any point:

If we know the center lines of the blades, the circulation distribution as well as the source and sink distribution, then with the aid of the plates I and II and using graphical integration one can get the effects:

- Effect of the mean value of the circulation.
- Effect of the mean value of the source and sink.
- Effect of the harmonics of the circulation, sources and sinks.

Adding all these effects the velocity at any point can be found, Figure 2.

If it happens that the point considered lies on the blade, a case of special importance, then consider the element of length δs about the station point P, Figure 3. The velocity calculated at the station point P on the blade center line is given by C_{st} . Let the velocity at the upper end at the lower sides of the blade to be given by C_{st_U} and C_{st_L} such that

$$C_{st_U} = C_{st} + \Delta C_y$$

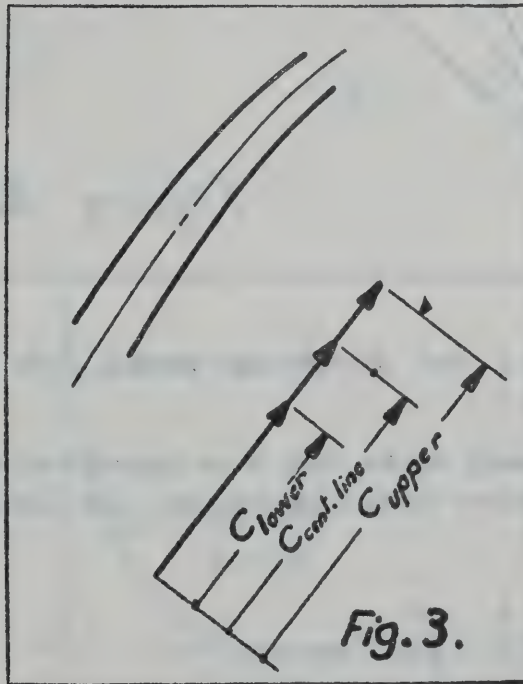
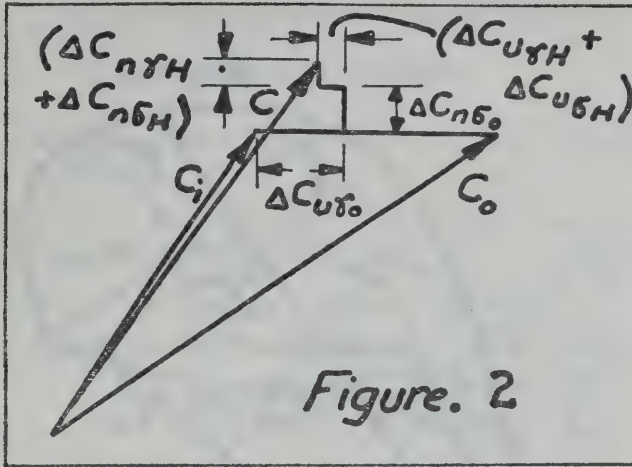
$$C_{st_L} = C_{st} - \Delta C_y$$

ΔC_y is to be calculated from the consideration of the circulation around the element about the station point P. This is given by $2 \Delta C_y \cdot \delta s$ and is equal to $\gamma_o \cdot t \cdot \delta s \cdot \sin \theta$ Hence

$$2 \Delta C_y \cdot \delta s = \gamma_o \cdot t \cdot \delta s \sin \theta$$

$$\text{i.e.} \quad \Delta C_y = \frac{1}{2} \cdot \gamma_o \cdot t \cdot \sin \theta \quad (7)$$

The hodograph representing the velocity at the blade surface is shown in Figure 4. Note that near the stagnation point the Ackeret cascade theory is not valid.



If the effect of the harmonics can be neglected, then the hodograph will take the shape shown in Figure 5.

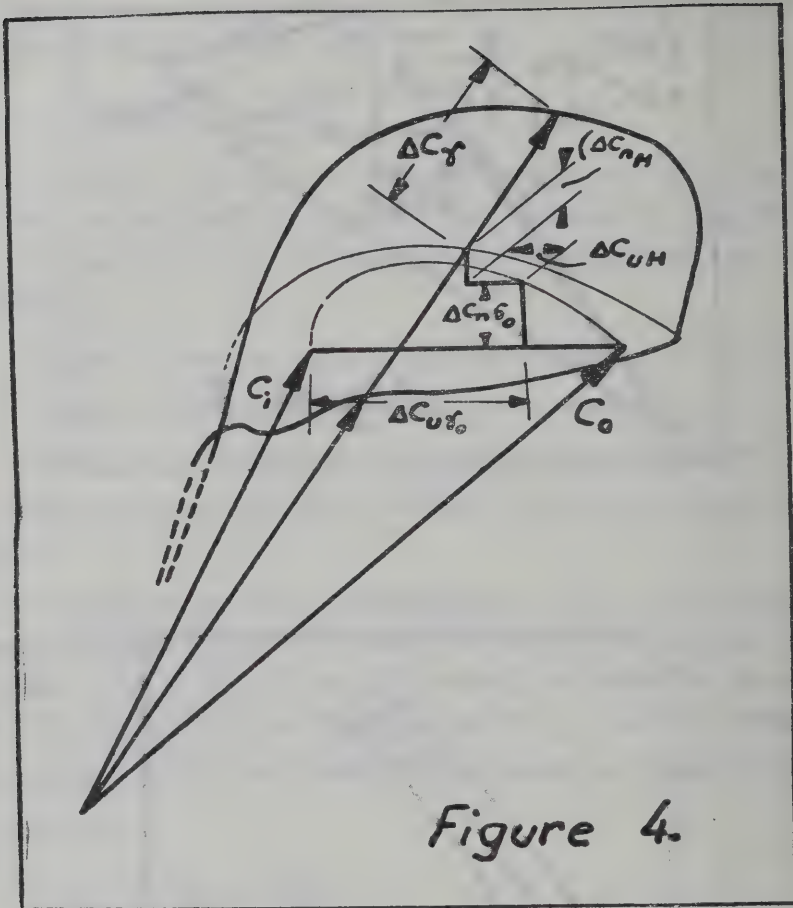


Figure 4.

The source and sink distribution derived from the thickness distribution :

It can be easily shown that, in an approximate way, the mean value of the source distribution (and the sink distribution) can be represented by

$$\sigma_o = \frac{1}{t} \cdot \frac{d}{dy} (c_{\text{cent. line}} \cdot \delta) \quad . \quad . \quad . \quad . \quad (8)$$

Generally the blades have rounded noses and hence σ_o will attend infinite value at the nose. To eliminate the difficulties in computation, we adopt another approximation as shown in Figure 6.

$C_{\text{cent. line}}$ = Velocity at blade center line

δ = Thickness of the blade.

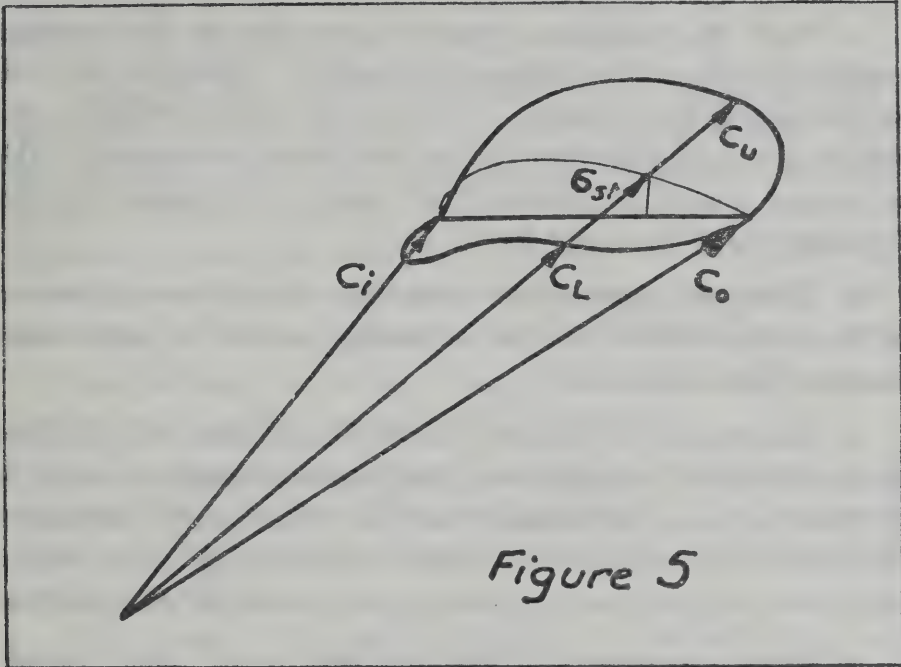


Figure 5

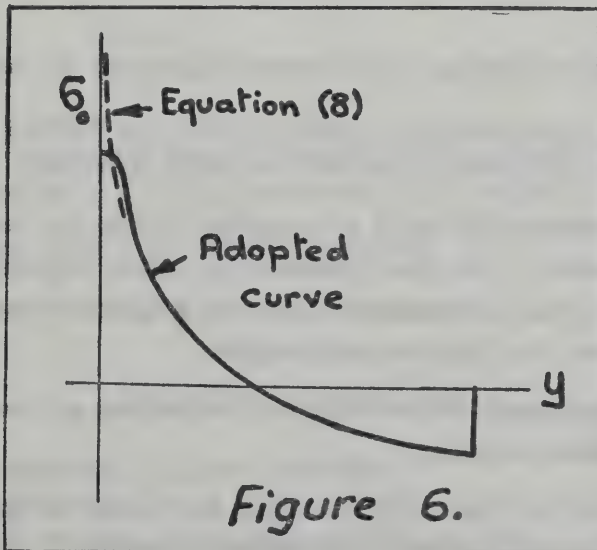


Figure 6.

DESIGN PROCEDURE

The following steps can be followed :

1. Begin by neglecting completely the effect of the harmonics as well as the effect of the thickness distribution. This gives rise to the one dimensional Euler theory. With the aid of the circulation distribution (assumed) the velocity at any station can be computed. Take two stream lines at a distance " t " apart to represent the first approximation of the center lines.

2. Taking the approximate center lines found from (1) one can find the source and sink distribution making use of the approximate velocities found also in (1).

3. The effect of the harmonics as well as the effect of the source distribution can be computed using the graphical integration with the aid of the curves shown in tables I and II. Hence we get the velocity distribution on the center line found before in (1). This velocity distribution will give rise to a better approximation of the center line.

4. Repeat procedure (3) with the new center line. Generally the new center line thus found needs no further corrections.

CASCADE DESIGN FOR MINIMUM MAXIMUM VELOCITY

To design a cascade for a certain output, the velocity distribution must be carefully selected to avoid local high velocities.

Flow separation is liable to take place when the pressure at the surface increases in the flow direction, *i.e.* when a positive pressure gradient takes place. A steeper pressure gradient is more likely to take place when local high velocities exist.

With compressible flow high local velocities are responsible for low critical Mach numbers.

For water flow local high velocities may cause cavitation as high velocity regions are generally low pressure regions.

From the above it is quite clear that the problem of minimising the maximum velocities is of vital importance. For potential flow the maximum velocity takes place at a point on the boundary. With the existence of the boundary layer the location of the maximum velocity is at a point just outside the boundary layer.

In the following a trial is presented to find approximately the best distribution of circulation or of the bounded vortices. By the best circulation distribution it is meant that distribution which gives rise to a minimum maximum velocity. The maximum velocity takes place at a point on the suction side. To have this maximum velocity a minimum means that the velocity all over the suction side is a constant. This latter statement is the basis of the following method.

The following restrictions, assumptions and simplifications are however made to solve the problem mathematically.

1. The discussion is restricted to lightly loaded cascades. This restriction limits the use of the results obtained. It is not however serious since lightly loaded cascades represent the conditions of many axial flow machines.

2. As a result of the above restriction the blades of the cascade are characterised by the slightly curved chamber line. We shall assume this chamber line to be a straight line parallel to the vector

$$\vec{C}_{\infty} = \frac{\vec{C}_i + \vec{C}_o}{2}$$

3. The angle between any velocity in the cascade plane and the cascade direction will be approximated to β_{∞} which is equal to the angle between $\vec{C}_{\infty}$ and the cascade direction.

4. The effect of all the harmonics is neglected,

CASE I.—VERY THIN BLADES

With reference to figure 7 we have :

$$\vec{C}_i = \vec{C}_o = \text{the mean velocity at inlet,}$$

$\vec{C}_0 = \vec{o a} =$ the mean velocity at outlet,

$t =$ the blade spacing, the unit length being the blade chord,

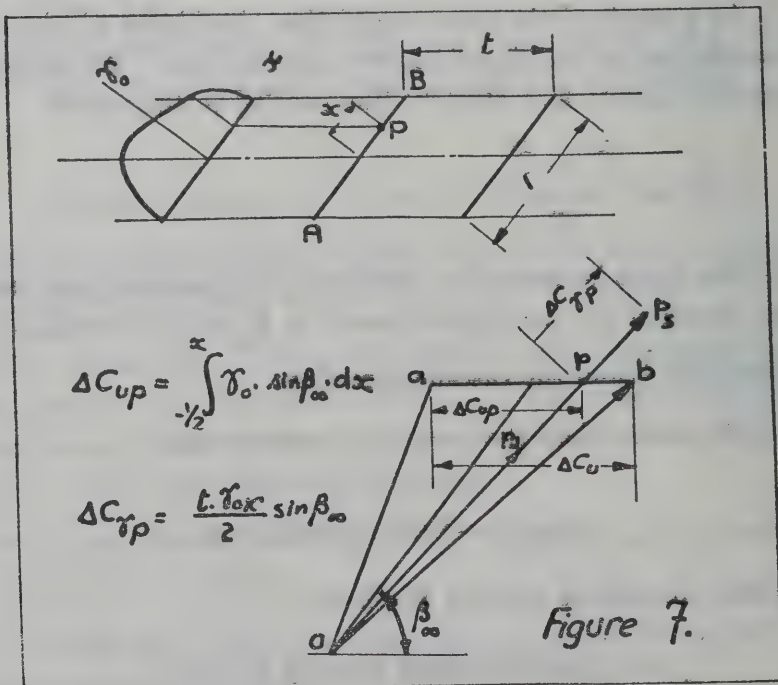
$\vec{o p} =$ the mean velocity at point "P"

$$= \frac{1}{2} (\vec{o p_s} + \vec{o p_d})$$

$\vec{o p_s} =$ The velocity at the point "P" and at the suction side of the blade.

$\vec{o p_d} =$ the velocity at the point "P" and at the pressure side of the blade.

On the same Figure 7 is shown the circulation distribution represented by its mean value γ_0 .



For the velocity vectors at P we have

$$\Delta C_u = -\frac{1}{2} \int^x \gamma_o \cdot \sin \beta_\infty \cdot dx$$

and
$$\Delta C_{\gamma p} = \frac{t \cdot \gamma_{ox} \cdot \sin \beta_\infty}{2}$$

The absolute value of the velocity vector $\vec{op}_s$ is given by ;

$$\begin{aligned} |\vec{op}_s| &= |\vec{C}_i| + |\vec{ap}| \cos \beta_\infty + |\vec{pp}_s| \\ &= C_i + \cos \beta_\infty \cdot -\frac{1}{2} \int^x \gamma_o \cdot \sin \beta_\infty dx + \frac{t \cdot \gamma_{ox} \cdot \sin \beta_\infty}{2} \end{aligned}$$

(Refer to the approximation mentioned on page 372).

To have the absolute value $|\vec{op}_s|$ a minimum the value of

$$\frac{d}{dx} |\vec{op}_s| \text{ must vanish, or } \frac{d}{dx} |\vec{op}_s| = 0.$$

$$\text{or } \frac{d}{dx} \left(c_i + \sin \beta_\infty \cdot \cos \beta_\infty \cdot -\frac{1}{2} \int^x \gamma_o \cdot dx + \frac{t \cdot \sin \beta_\infty}{2} \cdot \gamma_{ox} \right) = 0 \quad (9)$$

It is known that

$$\frac{d}{dx} \int^x f(\zeta) \cdot d\zeta = f(x)$$

Hence in our case

$$\frac{d}{dx} \cdot -\frac{1}{2} \int^x \gamma_o \cdot dx = \gamma_{ox}$$

Introduce this relation in equation 24 to get

$$\sin \beta_\infty \cdot \cos \beta_\infty \cdot \gamma_{ox} + \frac{t \cdot \sin \beta_\infty}{2} \cdot \frac{d(\gamma_{ox})}{dx} = 0$$

or
$$\frac{2 \cos \beta_{\infty}}{t} \cdot \gamma_{ox} + \frac{d \gamma_{ox}}{d x} = 0 \quad (10)$$

The solution of this differential equation is given by

$$\gamma_{ox} = A \cdot e^{-\frac{2 \cos \beta_{\infty}}{t} \cdot x} \quad (11)$$

The integration constant A is calculated from the "boundary" conditions and namely

$$\Delta C_u = \int_{-1/2}^{1/2} \gamma_{ox} \cdot \sin \beta_{\infty} \cdot dx$$

Introducing the value of γ_{ox} we get

$$\begin{aligned} \Delta C_u &= \sin \beta_{\infty} \int_{-1/2}^{1/2} A \cdot e^{-\frac{2 \cos \beta_{\infty}}{t} \cdot x} \cdot dx \\ &= -A \sin \beta_{\infty} \cdot \frac{t}{2 \cos \beta_{\infty}} \cdot \left[e^{-\frac{2 \cos \beta_{\infty}}{t} \cdot x} \right]_{-1/2}^{1/2} \\ &= -A \tan \beta_{\infty} \cdot \frac{t}{2} \left(e^{-\frac{\cos \beta_{\infty}}{t}} - e^{\frac{\cos \beta_{\infty}}{t}} \right) \\ &= A \cdot t \cdot \tan \beta_{\infty} \cdot \sinh \left(\frac{\cos \beta_{\infty}}{t} \right) \end{aligned}$$

Hence
$$A = \frac{\Delta C_u}{t} \cdot \frac{\cot \beta_{\infty}}{\sinh \left(\frac{\cos \beta_{\infty}}{t} \right)} \quad (11a)$$

For the evaluation of γ_{ox} the graphs in figures 8 and 9 are plotted ;

Figure 8: $A / \left(\frac{\Delta C_u}{t} \right)$ is plotted against β_{∞} with $\cos \beta_{\infty} / t$ as parameter,

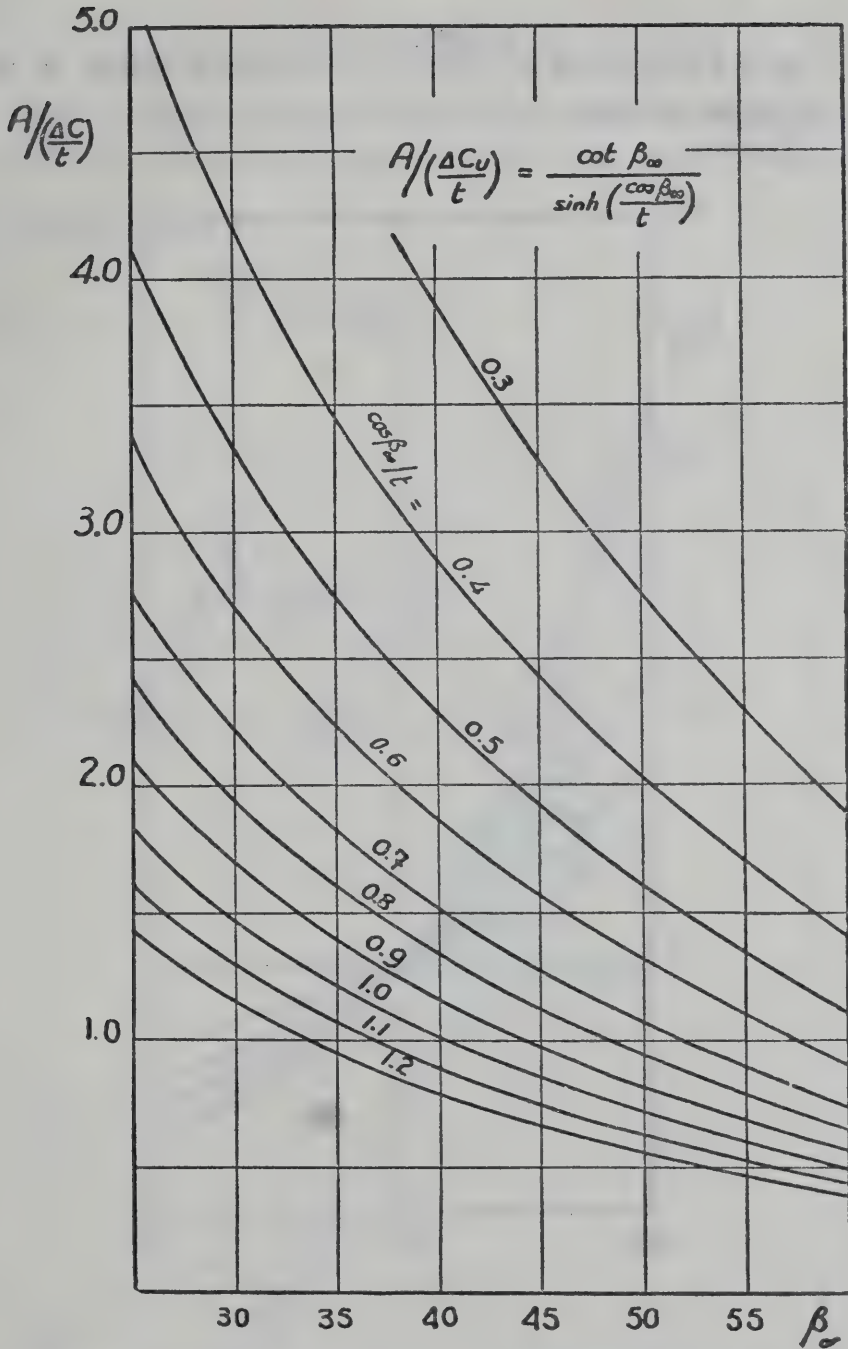
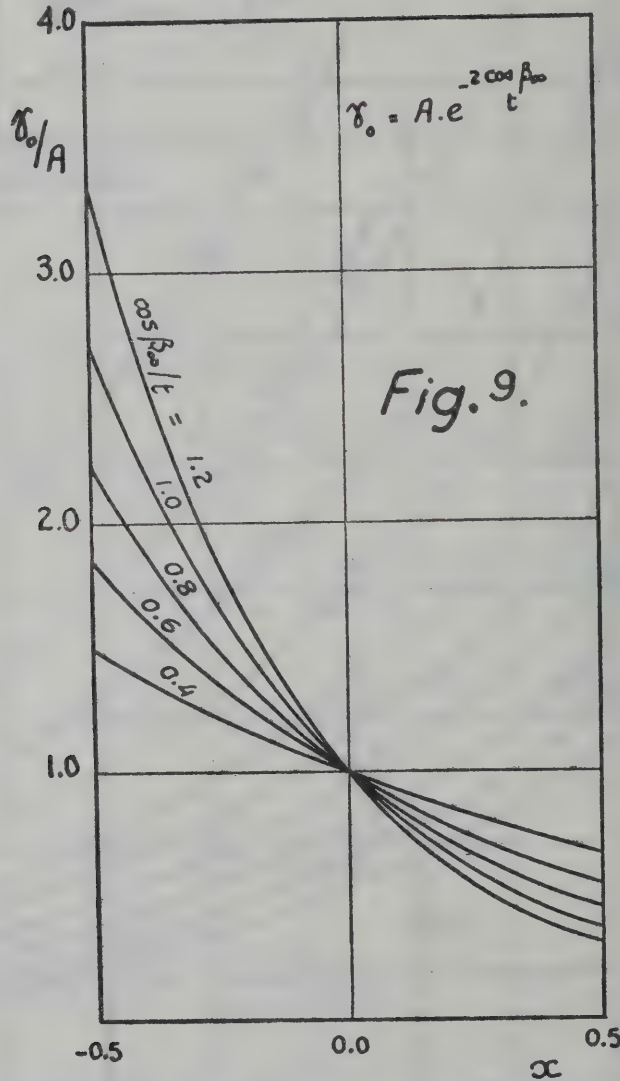


Fig. 8.

Figure 9: γ_{ox}/A or $e^{-\frac{2 \cos \beta_{\infty}}{t} x}$ is plotted against x with $(\cos \beta_{\infty}/t)$ as parameter.



The parameter used in the above plotting shall be called the covering ratio as it is equal to the ratio of the blade projection to the spacing of the blades.

CASE II.—EFFECT OF THE BLADE THICKNESS

With reference to Figure 10 the effect of the blade thickness is introduced by adding to the mean velocity at the point P the vector $\vec{P'P}$ which is equal to

$$\frac{\delta}{t} \cdot C_{\infty}$$

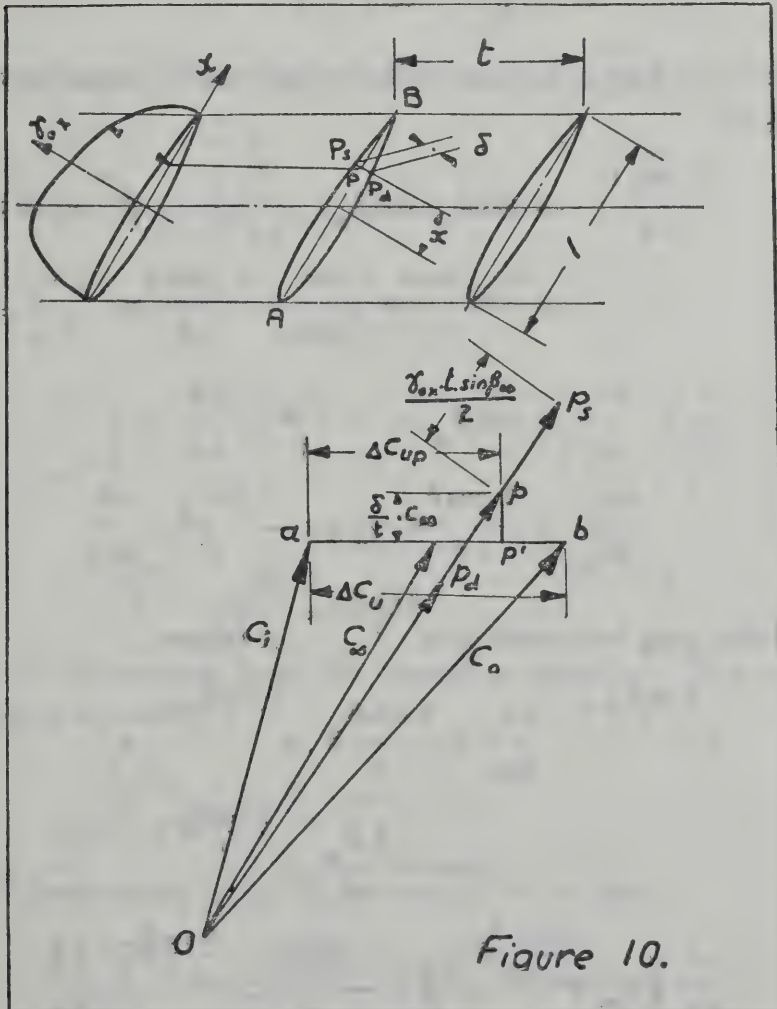


Figure 10.

where δ denotes the blade thickness at the point considred, δ can be generally expressed as a function of x .

The absolute value of the velocity at the suction side at point P_s is equal to

$$\begin{aligned} |\vec{op}_s| &= |\vec{C}_i| + |\vec{ap}'| \cdot \cos \beta_\infty + |\vec{p}'p| \sin \beta_\infty + |\vec{pp}_s| \\ &= C_i + \cos \beta_\infty \cdot \int_{-1/2}^x \gamma_o \cdot \sin \beta_\infty \cdot dx + \frac{\delta}{t} \cdot C_\infty \cdot \sin \beta_\infty \\ &\quad + \frac{t \cdot \sin \beta_\infty}{2} \cdot \gamma_o \end{aligned}$$

To have $|\vec{op}_s|$ a constant, its derivative with respect to x must vanish, or

$$\begin{aligned} \frac{d |\vec{op}_s|}{dx} &= \sin \beta_\infty \cdot \cos \beta_\infty \cdot \frac{d}{dx} \left(\int_{-1/2}^x \gamma_o \cdot dx \right) \\ &\quad + \frac{C_\infty \cdot \sin \beta_\infty}{t} \cdot \frac{d\delta}{dx} + \frac{t \cdot \sin \beta_\infty}{2} \cdot \frac{d\gamma_o}{dx} = 0 \end{aligned}$$

Hence

$$\begin{aligned} \frac{t}{2} \cdot \frac{d\gamma_o}{dx} + \cos \beta_\infty \cdot \gamma_o + \frac{C_\infty}{t} \cdot \frac{d\delta}{dx} &= 0 \\ \frac{d\gamma_o}{dx} + \frac{2 \cos \beta_\infty}{t} \gamma_o &= - \frac{2 C_\infty}{t^2} \cdot \frac{d\delta}{dx} \end{aligned}$$

Multiplying both sides by $e^{\frac{2 \cos \beta_\infty}{t} x}$, it follows

$$\begin{aligned} e^{\frac{2 \cos \beta_\infty}{t} x} \cdot \frac{d\gamma_o}{dx} + \frac{2 \cos \beta_\infty}{t} \cdot e^{\frac{2 \cos \beta_\infty}{t} x} \cdot \gamma_o \\ = - \frac{2 C_\infty}{t^2} \cdot e^{\frac{2 \cos \beta_\infty}{t} x} \cdot \frac{d\delta}{dx} \\ \frac{d}{dx} \left(\gamma_o \cdot e^{\frac{2 \cos \beta_\infty}{t} x} \right) = - \frac{2 C_\infty}{t^2} \cdot e^{\frac{2 \cos \beta_\infty}{t} x} \cdot \frac{d\delta}{dx} \end{aligned}$$

By Integration

$$\begin{aligned}
 \gamma_{ox} \cdot e^{\frac{2 \cos \beta_{\infty}}{t} x} &= -\frac{2 C_{\infty}}{t^2} \cdot \int e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{d \delta}{d x} \cdot d x \\
 &= -\frac{2 C_{\infty} \cdot \delta_{\max}}{t^2} \cdot \int e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{d (\delta / \delta_{\max})}{d x} \\
 &= -\frac{2 C_{\infty} \cdot \delta_{\max}}{t^2} \left(-\frac{1}{2} \int e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{d (\delta / \delta_{\max})}{d x} \right) + K
 \end{aligned}$$

K is the integration constant which can be written as

$$K = A + B$$

A is the constant given in equation (11a)

$$A = \frac{\Delta C_u}{t} \cdot \frac{\cot \beta_{\infty}}{\sinh \left(\frac{\cos \beta_{\infty}}{t} \right)}$$

$$\text{Hence } \gamma_{ox} = A \cdot e^{-\frac{2 \cos \beta_{\infty}}{t} x} + e^{-\frac{2 \cos \beta_{\infty}}{t} x} \left(B - \frac{2 C_{\infty}}{t} \cdot \frac{\delta_{\max}}{t} \cdot -\frac{1}{2} \int e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{d (\delta / \delta_{\max})}{d x} d x \right) \quad (12)$$

B is determined from the boundary condition. This can be formulated as follows

$$-\frac{1}{2} \int_{-1/2}^{1/2} \gamma_{ox} \cdot \sin \beta_{\infty} \cdot dx = \Delta C_u$$

With the above choice of the value of A we have

$$-\frac{1}{2} \int_{-1/2}^{1/2} A \cdot e^{-\frac{2 \cos \beta_{\infty}}{t} x} \cdot \sin \beta_{\infty} \cdot dx = 0$$

It follows that

$$-\frac{1}{2} \int_{-1/2}^{1/2} e^{-\frac{2 \cos \beta_{\infty}}{t} x} \cdot \left(B - \frac{2 C_{\infty}}{t} \cdot \frac{\delta_{\max}}{t} \cdot \int_{-1/2}^x e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{d(\delta/\delta_{\max})}{dx} \cdot dx \right) \cdot \sin \beta_{\infty} \cdot dx = 0$$

As $\sin \beta_{\infty}$ and $\frac{2 C_{\infty}}{t} \cdot \frac{\delta_{\max}}{t}$ are constants, this integral equation can be simplified to

$$\left. \begin{aligned} & -\frac{1}{2} \int_{-1/2}^{1/2} e^{-\frac{2 \cos \beta_{\infty}}{t} x} \left(B' - \int_{-1/2}^x e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{d(\delta/\delta_{\max})}{dx} \cdot dx \right) dx = 0 \end{aligned} \right\} \quad (13)$$

$$\text{where } B = \frac{2 C_{\infty}}{t} \cdot \frac{\delta_{\max}}{t} \cdot B' \quad (13a)$$

We try now to get the value of the integral in equation 13.

$$\begin{aligned} I_1 &= -\frac{1}{2} \int_{-1/2}^{1/2} B' \cdot e^{-\frac{2 \cos \beta_{\infty}}{t} x} \cdot dx \\ &= -B' \cdot \frac{t}{2 \cos \beta_{\infty}} \cdot \left[e^{-\frac{2 \cos \beta_{\infty}}{t} x} \cdot x \right]_{-1/2}^{1/2} \\ &= -B' \cdot \frac{t}{2 \cos \beta_{\infty}} \left(e^{-\frac{\cos \beta_{\infty}}{t}} - e^{\frac{\cos \beta_{\infty}}{t}} \right) \\ &= B' \cdot \frac{t}{\cos \beta_{\infty}} \cdot \sinh \left(\frac{t}{\cos \beta_{\infty}} \right) \\ &\quad - \frac{1}{2} \int_{-1/2}^x e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{d(\delta/\delta_{\max})}{dx} \cdot dx \end{aligned}$$

$$\begin{aligned}
 &= \left| e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{\delta}{\delta_{\max}^{-1/2}} \right|_{-1/2}^x - \int_{-1/2}^x \frac{2 \cos \beta_{\infty}}{t} \cdot \\
 &\quad e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{\delta}{\delta_{\max}} \cdot dx \\
 &= e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{\delta}{\delta_{\max}} - \frac{2 \cos \beta_{\infty}}{t} \cdot g(x)
 \end{aligned}$$

where $g(x) = \int_{-1/2}^x e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{\delta}{\delta_{\max}} \cdot dx \quad (14)$

$$\begin{aligned}
 I_2 &= \int_{-1/2}^x e^{-\frac{2 \cos \beta_{\infty}}{t} x} \left(\int_{-1/2}^x e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{d(\delta/\delta_{\max})}{dx} \cdot dx \right) \cdot dx \\
 &= \int_{-1/2}^{1/2} e^{-\frac{2 \cos \beta_{\infty}}{t} x} \cdot \left(e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{\delta}{\delta_{\max}} - \frac{2 \cos \beta_{\infty}}{t} \cdot g(x) \right) dx \\
 &= \int_{-1/2}^{1/2} \left(\frac{\delta}{\delta_{\max}} - \frac{2 \cos \beta_{\infty}}{t} \cdot e^{-\frac{2 \cos \beta_{\infty}}{t} x} \cdot g(x) \right) dx \\
 &= D - \frac{2 \cos \beta_{\infty}}{t} \cdot \int_{-1/2}^{1/2} e^{-\frac{2 \cos \beta_{\infty}}{t} x} \cdot g(x) \cdot dx
 \end{aligned}$$

where $D = \int_{-1/2}^{1/2} \frac{\delta}{\delta_{\max}} \cdot dx$

Rewrite I_2 in the form

$$\begin{aligned}
 I_2 &= D - \frac{2 \cos \beta_{\infty}}{t} \left(-g(x) \cdot e^{-\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{t}{2 \cos \beta_{\infty}} \right) \Big|_{-1/2}^{1/2} \\
 &\quad + \frac{t}{2 \cos \beta_{\infty}} \int_{-1/2}^{1/2} e^{-\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{d(g(x))}{dx} dx
 \end{aligned}$$

$$\begin{aligned}
 &= D + g\left(\frac{1}{2}\right) \cdot e^{-\frac{\cos \beta_{\infty}}{t}} - \\
 &\quad - \int_{-1/2}^{1/2} e^{-\frac{2 \cos \beta_{\infty}}{t} x} \cdot e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{\delta}{\delta_{\max}} \cdot dx \\
 &= D + g\left(\frac{1}{2}\right) \cdot e^{-\frac{\cos \beta_{\infty}}{t}} - \int_{-1/2}^{1/2} \frac{\delta}{\delta_{\max}} dx \\
 &= D + g\left(\frac{1}{2}\right) \cdot e^{-\frac{\cos \beta_{\infty}}{t}} - D \\
 &= g\left(\frac{1}{2}\right) \cdot e^{-\frac{\cos \beta_{\infty}}{t}}
 \end{aligned}$$

Introducing the values of I_1 and I_2 in equation 13 we get

$$I_1 = I_2$$

$$\begin{aligned}
 \therefore B' \cdot \frac{t}{\cos \beta_{\infty}} \cdot \sinh\left(\frac{\cos \beta_{\infty}}{t}\right) &= g\left(\frac{1}{2}\right) \cdot e^{-\frac{\cos \beta_{\infty}}{t}} \\
 \therefore B' &= g\left(\frac{1}{2}\right) \cdot \frac{\cos \beta_{\infty}}{t} \cdot e^{-\frac{\cos \beta_{\infty}}{t}} / \sinh\left(\frac{\cos \beta_{\infty}}{t}\right) \\
 \therefore B &= \left. \begin{aligned} &\frac{2 C_{\infty}}{t} \frac{\delta_{\max}}{t} \cdot g\left(\frac{1}{2}\right) \cdot \frac{\cos \beta_{\infty}}{t} \cdot \\ &e^{-\frac{\cos \beta_{\infty}}{t}} / \sinh\left(\frac{\cos \beta_{\infty}}{t}\right) \end{aligned} \right\} \quad (15)
 \end{aligned}$$

This equation together with equation 14

$$g(x) = \int_{-1/2}^x e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{\delta}{\delta_{\max}} \cdot dx \quad (14)$$

determines the value of B in equation 12 for the circulation distribution γ_{ox} .

$$\begin{aligned}
 \gamma_{ox} &= A \cdot e^{-\frac{2 \cos \beta_{\infty}}{t} x} + e^{-\frac{2 \cos \beta_{\infty}}{t} x} \left(B - \right. \\
 &\quad \left. 2 \frac{C_{\infty}}{t} \cdot \frac{\delta_{\max}}{t} \cdot \int_{-1/2}^x e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{d(\delta/\delta_{\max})}{dx} \cdot dx \right) \\
 &= (A + B) \cdot e^{-\frac{2 \cos \beta_{\infty}}{t} x} - 2 \frac{C_{\infty}}{t} \cdot \frac{\delta_{\max}}{t} \cdot e^{-\frac{2 \cos \beta_{\infty}}{t} x} \cdot \\
 &\quad \left(e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{\delta}{\delta_{\max}} - \frac{2 \cos \beta_{\infty}}{t} \cdot \int_{-1/2}^x e^{\frac{2 \cos \beta_{\infty}}{t} x} \cdot \frac{\delta}{\delta_{\max}} dx \right) \\
 &= (A + B) \cdot e^{-\frac{2 \cos \beta_{\infty}}{t} x} - 2 \frac{C_{\infty}}{t} \cdot \frac{\delta_{\max}}{t} \cdot \\
 &\quad \left(\frac{\delta}{\delta_{\max}} - \frac{2 \cos \beta_{\infty}}{t} \cdot e^{-\frac{2 \cos \beta_{\infty}}{t} x} \cdot g(x) \right) \\
 &= (A + B) \cdot e^{-\frac{2 \cos \beta_{\infty}}{t} x} - 2 \frac{C_{\infty}}{t} \cdot \frac{\delta_{\max}}{t} \cdot h(x) \quad (16)
 \end{aligned}$$

where $h(x) = \frac{\delta}{\delta_{\max}} - \frac{2 \cos \beta_{\infty}}{t} \cdot e^{-\frac{2 \cos \beta_{\infty}}{t} x} \cdot g(x)$

It is quite clear that $g(x)$, B and $h(x)$ vary with the blade form used. In the following the discussion is restricted to the blade form used for the four—and five—digit NACA profiles. The corresponding thickness distribution is given by

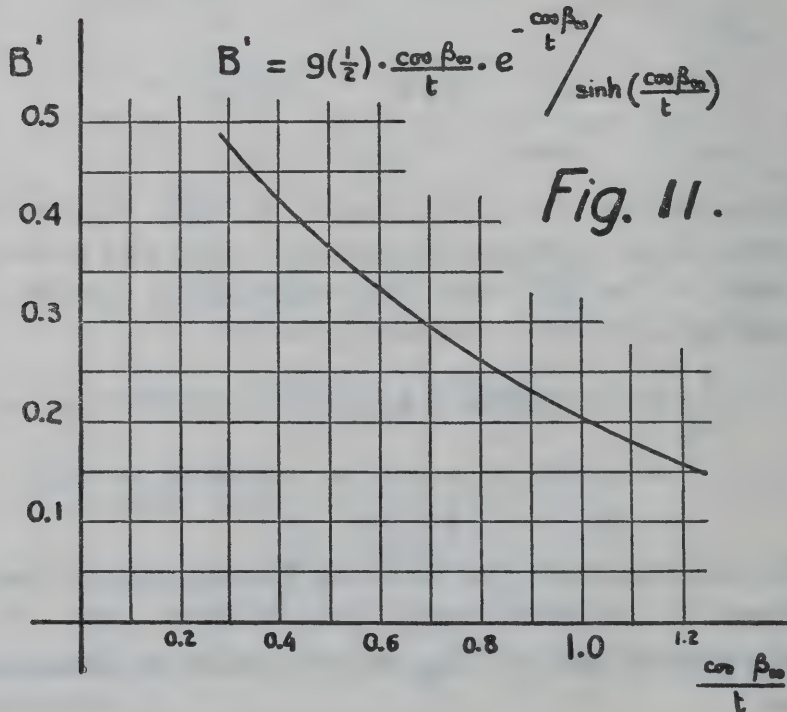
$$\begin{aligned}
 \frac{\delta}{\delta_{\max}} &= 0.1 (0.29690 \sqrt{x + 0.5} - 0.12600 (x + 0.5) \\
 &\quad - 0.35160 (x + 0.5)^2 + 0.28430 (x + 0.5)^3 \\
 &\quad - 0.10150 (x + 0.5)^4)
 \end{aligned}$$

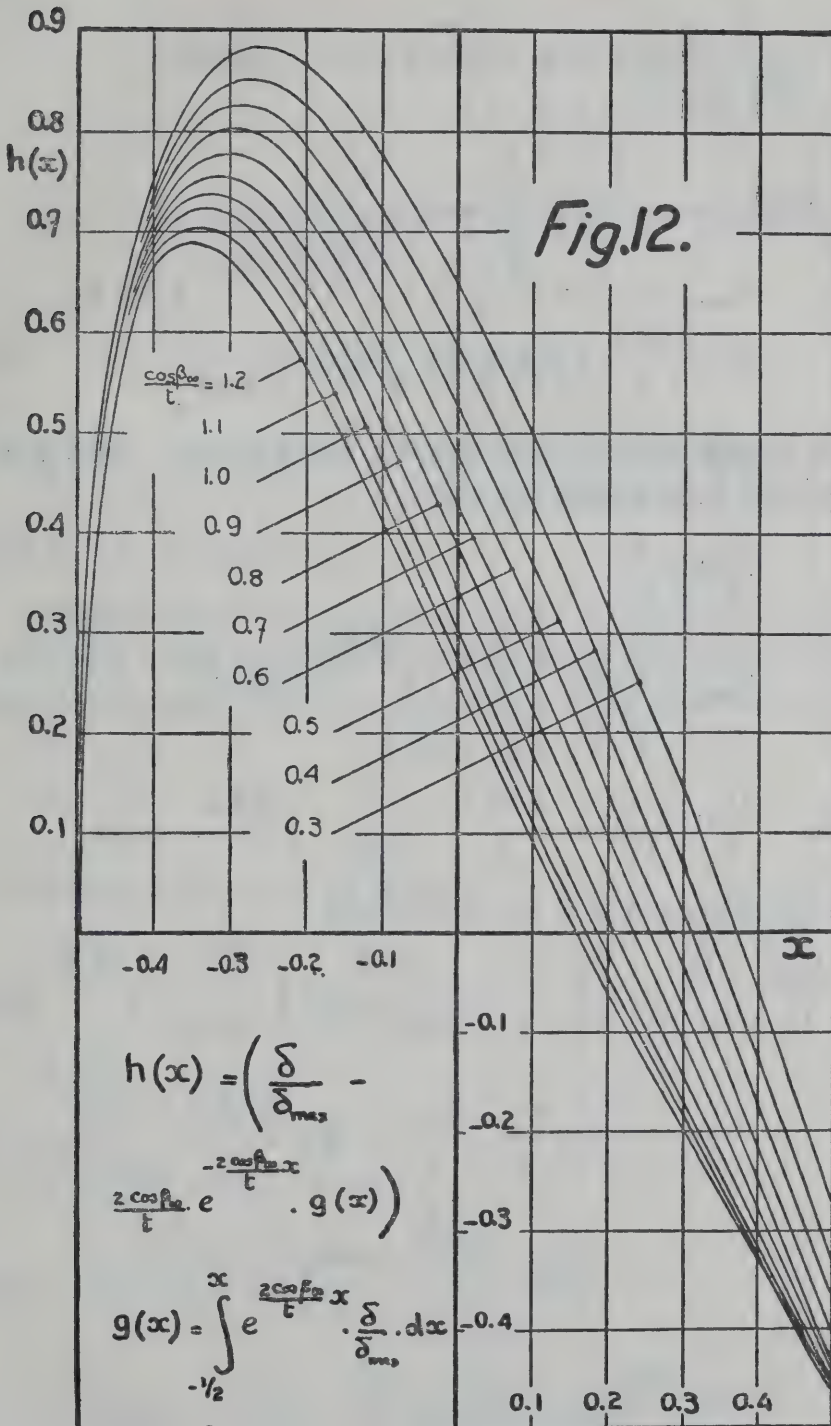
For this blade form the maximum thickness occurs nearly at a distance 0.3 the chord length from the blade nose. In the following table the value of $\delta/\delta_{\max}$ is given against the distance from the nose.

TABLE

% age distance from nose : 0.0 1.25 2.5 5 10						
% age $\delta/\delta_{\max}$: 0.0 31.56 43.56 59.24 78.04						
15	20	30	40	50	60	70
89.10	96.64	100.0	96.64	88.14	76.06	61.06
80	90	100				
43.74	24.14	2.1				

For this blade form the calculations for B' and $h(x)$ are made and they are shown in Figures 11 and 12. Figure 11 shows the relation between B' and $\cos \beta_{\infty}/t$. Figure 12 shows the relation between $h(x)$ and x with the covering ratio $\frac{\cos \beta_{\infty}}{t}$ as parameter.





THE VALUE OF THE MAXIMUM VELOCITY

It is clear that at $x = -\frac{1}{2}$

$$h(x) = 0$$

Mence the velocity at the suction sede is given by

$$\begin{aligned} C_{\max} C &= C_i + \frac{A+B}{2} e^{-\frac{2 \cos \beta_{\infty}}{t} \cdot (-\frac{1}{2})} \cdot t \cdot \sin \beta_{\infty} \\ &= C_i + \frac{A+B}{2} \cdot e^{\frac{\cos \beta_{\infty}}{t}} \cdot t \cdot \sin \beta_{\infty} \end{aligned}$$

It is quite interesting to compare this maximum velocity with the velocity C_o by getting the ratio

$$\frac{C_{\max} - C_i}{C_o - C_i}$$

$$C_{\max} - C_i = \frac{A+B}{2} \cdot e^{\frac{\cos \beta_{\infty}}{t}} \cdot t \cdot \sin \beta_{\infty}$$

$$C_o - C_i = \Delta C_u \cdot \cos \beta_{\infty}$$

$$\text{Hence } \frac{C_{\max} - C_i}{C_o - C_i} = \frac{A+B}{2} \cdot \frac{t}{\Delta C_u} \cdot e^{\frac{\cos \beta_{\infty}}{t}} \cdot \tan \beta_{\infty}$$

Inserting the values of A and B it follows

$$\begin{aligned} \frac{C_{\max} - C_i}{C_o - C_i} &= \frac{1}{2} \left(\frac{\Delta C_u}{t} \cdot \frac{\cot \beta_{\infty}}{\sinh \left(\frac{\cos \beta_{\infty}}{t} \right)} \right) \cdot \frac{t}{\Delta C_u} \cdot e^{\frac{\cos \beta_{\infty}}{t}} \tan \beta_{\infty} \\ &+ \frac{1}{2} \left(2 \frac{C_{\infty}}{t} \cdot \frac{\beta_{\max}}{t} \cdot g \left(\frac{1}{2} \right) \cdot \frac{\cos \beta_{\infty}}{t} \cdot \frac{e^{\frac{\cos \beta_{\infty}}{t}}}{\sinh \left(\frac{\cos \beta_{\infty}}{t} \right)} \right) \\ &\quad \frac{t}{\Delta C_u} \cdot e^{\frac{\cos \beta_{\infty}}{t}} \cdot \tan \beta_{\infty} \\ &= \frac{1}{2} \cdot \frac{e^{\frac{\cos \beta_{\infty}}{t}}}{\sinh \left(\frac{\cos \beta_{\infty}}{t} \right)} \end{aligned}$$

$$\begin{aligned}
 & + \frac{C_{\infty}}{\Delta C_U} \cdot \frac{\delta_{\max}}{t} \cdot \frac{\cos \beta_{\infty}}{t} \cdot g\left(\frac{1}{2}\right) \cdot \frac{\tan \beta_{\infty}}{\sinh\left(\frac{\cos \beta_{\infty}}{t}\right)} \\
 & = \frac{1}{2} \frac{e^{\frac{\cos \beta_{\infty}}{t}}}{\sinh\left(\frac{\cos \beta_{\infty}}{t}\right)} + \frac{C_{\infty}}{\Delta C_U} \cdot \frac{\delta_{\max}}{t} \cdot \tan \beta_{\infty} \cdot \phi\left(\frac{\cos \beta_{\infty}}{t}\right)
 \end{aligned}$$

where $\phi\left(\frac{\cos \beta_{\infty}}{t}\right) = \frac{\cos \beta_{\infty}}{t} \cdot g\left(\frac{1}{2}\right) / \sinh\left(\frac{\cos \beta_{\infty}}{t}\right)$

The curves for $e^{\frac{\cos \beta_{\infty}}{t}} / 2 \sinh\left(\frac{\cos \beta_{\infty}}{t}\right)$ and $\phi\left(\frac{\cos \beta_{\infty}}{t}\right)$ are shown in Figures 13 and 14.

Example: The velocity diagram shown in Figure 16 represent the inlet and outlet velocities it is required to design the cascade where the maximum velocity is limited to 1.05 the outlet velocity.

Take $\frac{\delta_{\max}}{t} = 0.05$.

Refering to Figure 28 we have

$$C_i = 1.342 C_n$$

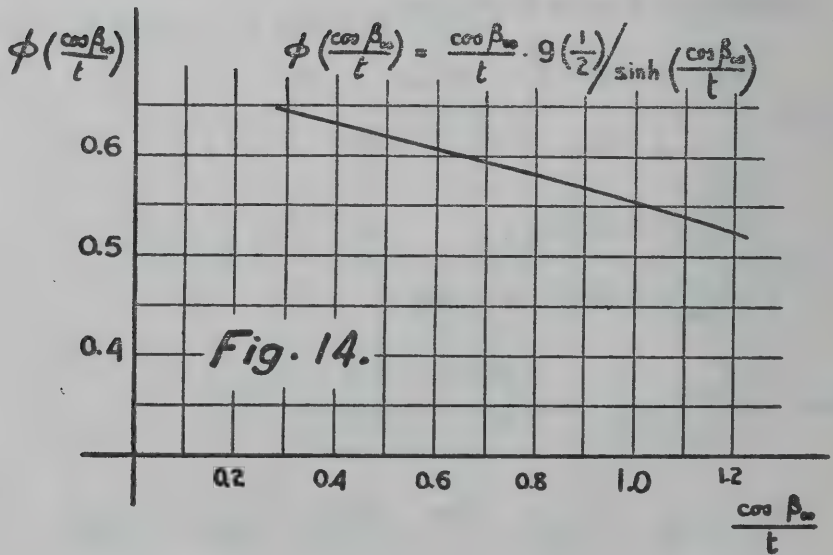
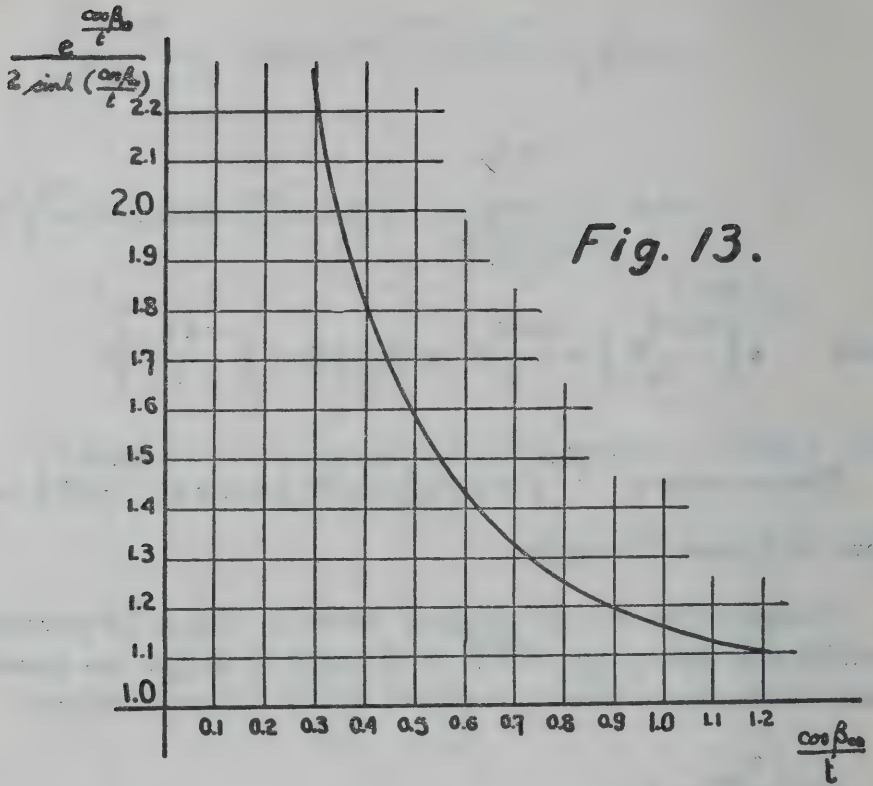
$$C_o = 1.487 C_n$$

$$C_{\infty} = 1.414 C_n$$

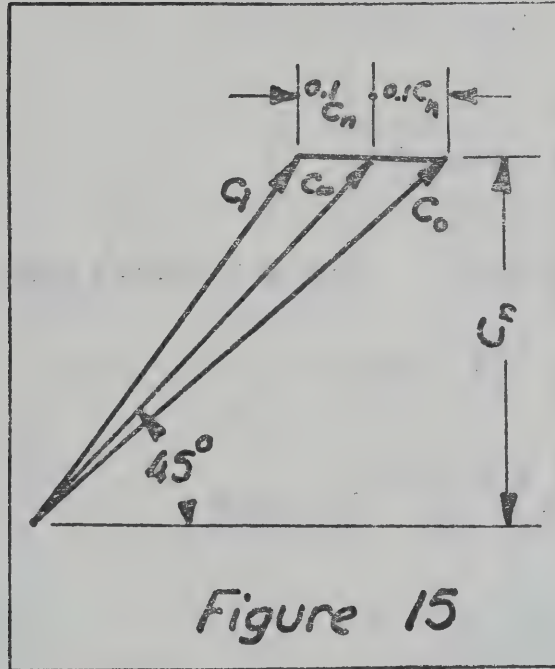
$$C_{\max} = 1.05 C_o = 1.561 C_n$$

Hence $\frac{C_{\max} - C_i}{C_o - C_i} = \frac{1.561 - 1.342}{1.487 - 1.342} = 1.51$

As $\tan \beta_{\infty} \cdot \frac{C_{\infty}}{\Delta C_u} \cdot \frac{\delta_{\max}}{t} = \frac{1.414}{0.2} \times 1.05 = 0.3535$



$$\frac{C_{\max} - C_i}{C_o - C_i} = \frac{1}{2} \cdot \frac{e^{\frac{\cos \beta_{\infty}}{t}}}{\sinh\left(\frac{\cos \beta_{\infty}}{t}\right)} + 0.3535 \phi\left(\frac{\cos \beta_{\infty}}{t}\right)$$



$\frac{C_{\max} - C_i}{C_o - C_i}$ varies with $\frac{\cos \beta_{\infty}}{t}$ as shown in the following table

$\cos \beta_{\infty} / t$	0.5	0.6	0.7	0.8	0.9	1.0
$\frac{C_{\max} - C_i}{C_o - C_i}$	1.701	1.647	1.535	1.46	1.399	1.353

From this table the value of the covering ratio corresponding to the maimum velocity of 1.05 C_o is equal to 0.73.

Take the covering ratio = 0.8

$$\text{Hence } A = 1.126 \frac{\Delta C_u}{t} \text{ (Figure 21, Table 1 App.)}$$

$$= 1.126 \frac{0.2 C_n}{t}$$

$$= 0.2252 \frac{C_n}{t}$$

$$B' = 0.2612 \quad (\text{Figure 11, Table 3 App.})$$

$$B = \frac{2 C_\infty}{t} \cdot \frac{\delta_{\max}}{t} \cdot B'$$

$$= \frac{2 \times 1.414 C_n}{t} \times 0.05 B'$$

$$= 0.0369 \frac{C_n}{t}$$

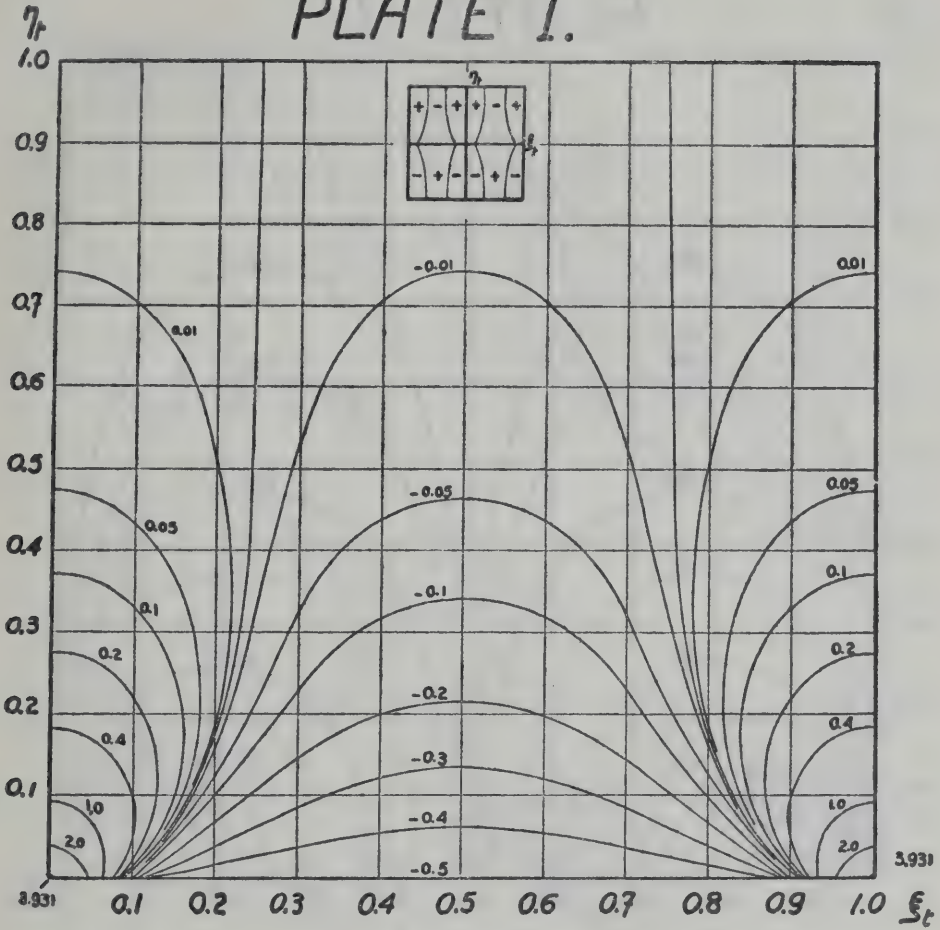
$$\text{Hence } A + B = 0.2621 \frac{C_n}{t}$$

$$\Delta s \quad 2 \frac{C_\infty}{t} \cdot \frac{\delta_{\max}}{t} = 0.1414 \frac{C_n}{t}$$

It follows

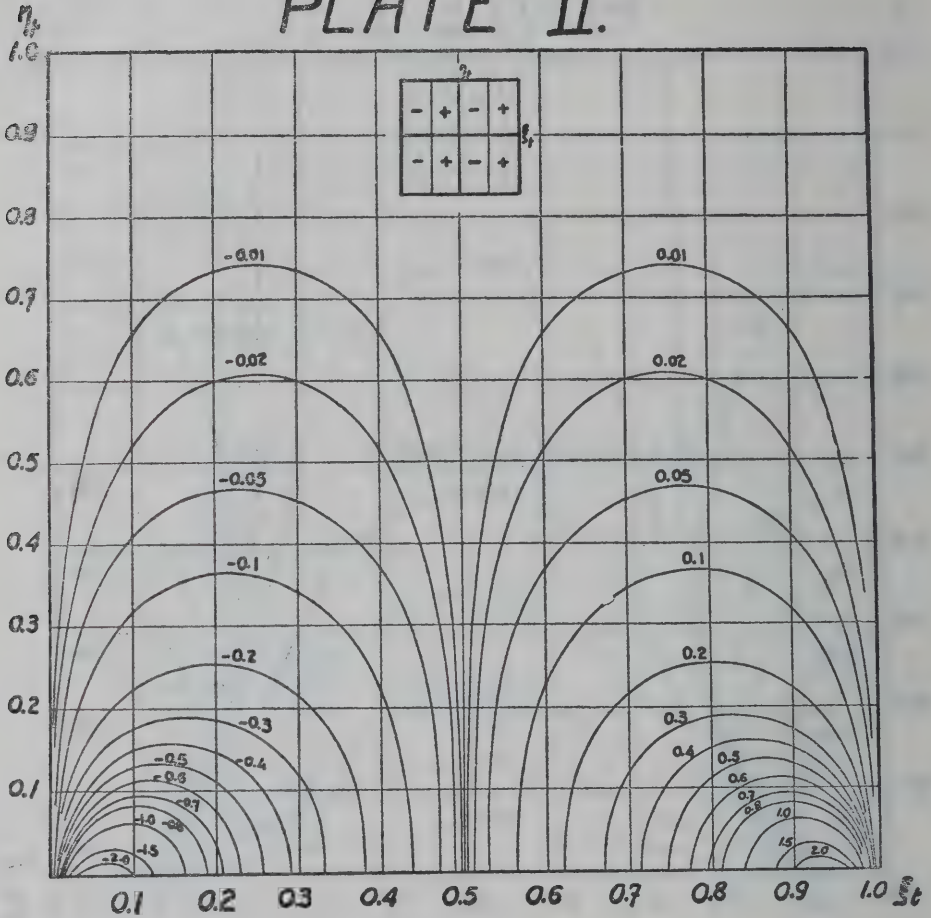
$$y_o = (0.2621 e^{\frac{2 \cos \beta_\infty x}{t}} - 0.1414 h(x)) \frac{C_n}{t}$$

PLATE I.



$$\frac{1}{\gamma_0} \frac{du_x}{dy} = \sum_{n=1}^{\infty} e^{-\left(\frac{n}{2}\right)^2} \cdot e^{-2\pi n \eta_t} \cdot \cos(2\pi n \xi_t) = \frac{1}{6} \cdot \frac{d\psi_0}{dy}$$

PLATE II.



$$\frac{1}{\tau_0} \cdot \frac{dU_T}{dy} = - \sum_{n=1}^{\infty} e^{-\left(\frac{n}{2}\right)^2} e^{-2\pi n \eta} \cdot \sin(2\pi n \xi) = - \frac{1}{6_0} \cdot \frac{dU_6}{dy}$$

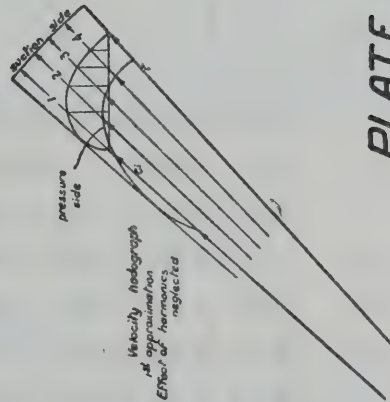
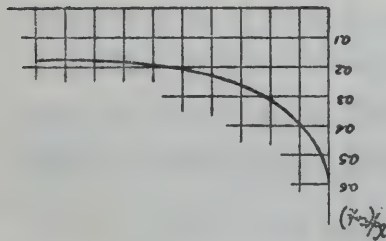


PLATE III.

With reference to Figures 9 and 12 (or tables 2 and 4 App.) the variation of γ_0 with x is as follows

x	-0.5	-0.4	-0.3	-0.2	-0.1	-0.0
γ_0	0.5822	0.3986	0.3157	0.2683	0.2303	0.2175
	0.1	0.2	0.3	0.4	0.5	
	0.1921	0.1838	0.1819	0.1779	0.1803	

The design is as in plate III.

TABLE 1

$$A / \frac{\Delta C_u}{t} = \frac{\cot \beta_\infty}{\sinh \left(\frac{\cos \beta_\infty}{t} \right)}$$

β_∞ $\frac{\cos \beta_\infty}{t}$	25°	30	35	40	45	50	55	60	65
1.2	1.4215	1.1475	0.9465	0.7895	0.6627	0.5558	0.4634	0.3823	0.309
1.1	1.605	1.2965	1.069	0.8925	0.749	0.6281	0.5239	0.4318	0.349
1.0	1.825	1.4725	1.2155	1.014	0.851	0.726	0.5952	0.4908	0.3968
0.9	2.089	1.686	1.3905	1.1605	0.9735	0.817	0.6812	0.5620	0.454
0.8	2.415	1.950	1.608	1.3425	1.126	0.9452	0.7885	0.650	0.5252
0.7	2.732	2.221	1.818	1.5175	1.274	1.069	0.8918	0.7352	0.594
0.6	3.371	2.723	2.246	1.8735	1.572	1.319	1.101	0.9075	0.7335
0.5	4.115	3.323	2.740	2.287	1.9185	1.6105	1.343	1.1075	0.895
0.4	5.221	4.216	3.475	2.901	2.434	2.042	1.704	1.405	1.1355
0.3	7.045	5.477	4.515	3.912	3.283	2.755	2.298	1.895	1.5315

TABLE 2

$$e^{-\frac{2 \cos \beta_{\infty}}{t} x}$$

$\frac{\cos \beta_{\infty}}{t} \backslash x$	1.2	1.1	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3
—0.5	3.3201	3.004	2.7183	2.460	2.222	2.014	1.822	1.6487	1.4918	1.3499
—0.4	2.612	2.411	2.226	2.054	1.869	1.7507	1.6161	1.4918	1.3771	1.2712
—0.3	2.054	1.935	1.822	1.7160	1.6161	1.5220	1.4333	1.3499	1.2712	1.1972
—0.2	1.6161	1.5527	1.4918	1.4332	1.7771	1.3231	1.2712	1.2214	1.1735	1.1275
—0.1	1.2712	1.2461	1.2214	1.1972	1.1735	1.1503	1.1275	1.1052	1.0833	1.0618
0.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.1	0.7866	0.8025	0.8187	0.8353	0.8521	0.8694	0.8869	0.9048	0.9231	0.9418
0.2	0.6188	0.644	0.6703	0.6977	0.7261	0.7558	0.7866	0.8187	0.8521	0.8869
0.3	0.4868	0.5169	0.5488	0.5827	0.6188	0.657	0.6977	0.7408	0.7866	0.8353
0.4	0.3829	0.4148	0.4493	0.4868	0.5273	0.5712	0.6188	0.6703	0.7261	0.7866
0.5	0.3012	0.3329	0.3679	0.4066	0.4493	0.4966	0.5488	0.6065	0.6703	0.7408

TABLE 3

$$B' = g\left(\frac{1}{2}\right) \cdot \frac{\cos \beta_{\infty}}{t} \cdot e^{-\frac{\cos \beta_{\infty}}{t}} / \sinh\left(\frac{\cos \beta_{\infty}}{t}\right)$$

$\frac{\cos \beta_{\infty}}{t}$	1.2	1.1	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3
B'	0.1571	0.1803	0.2042	0.2313	0.2613	0.2949	0.3331	0.3758	0.4238	0.477

TABLE 4

$$h(x) = \frac{\delta}{\delta_{\max}} - \frac{2 \cos \beta_{\infty}}{t} \cdot e^{-\frac{2 \cos \beta_{\infty}}{t} x} \cdot g(x)$$

$\frac{\cos \beta_{\infty}}{t}$ x	1.2	1.1	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3
—0.5	0	0	0	0	0	0	0	0	0	0
—0.4	0.6637	0.6724	0.6831	0.6886	0.6946	0.7075	0.7189	0.7258	0.7348	0.7489
—0.3	0.6780	0.6940	0.7163	0.7342	0.7538	0.7785	0.8032	0.8244	0.8481	0.8762
—0.2	0.5692	0.593	0.6226	0.6491	0.6810	0.7153	0.7511	0.785	0.826	0.8654
—0.1	0.4125	0.440	0.4732	0.5115	0.5449	0.5858	0.6299	0.6735	0.7220	0.7778
0.0	0.2510	0.2774	0.296	0.3465	0.3865	0.4315	0.4804	0.530	0.5864	0.652
0.1	0.0915	0.1148	0.1342	0.1802	0.2208	0.2652	0.316	0.367	0.4285	0.4998
0.2	—0.0626	—0.045	—0.0285	0.011	0.046	0.0878	0.135	0.195	0.2485	0.3220
0.3	—0.1952	—0.1835	—0.1708	—0.1415	—0.121	—0.0762	—0.0334	0.0212	..0727	0.1450
0.4	—0.3348	—0.3292	—0.322	—0.3012	—0.281	—0.2538	—0.219	—0.1723	—0.1257	—0.0795
0.5	—0.4568	—0.4610	—0.4575	—0.4538	—0.443	—0.4264	0.401	0.3708	—0.3273	—0.2699

TABLE 5

$$\phi \left(\frac{\cos \beta_{\infty}}{t} \right) = \frac{\cos \beta_{\infty}}{t} \cdot g \left(\frac{1}{2} \right) / \sinh \left(\frac{\cos \beta_{\infty}}{t} \right)$$

$\frac{\cos \beta_{\infty}}{t}$	1.2	1.1	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3
ϕ	0.525	0.5418	0.555	0.569	0.5815	0.5945	0.607	0.620	0.6324	0.6445

TABLE 6

$$e^{\frac{\cos \beta_{\infty}}{t}} / 2 \sinh \left(\frac{\cos \beta_{\infty}}{t} \right)$$

$\frac{\cos \beta_{\infty}}{t}$	1.2	1.1	1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3
—	1.105	1.125	1.1565	1.198	1.253	1.327	1.432	1.582	1.816	2217

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ELECTRICAL ENGINEERING

THEORY OF THE SQUIRREL — CAGE INDUCTION MOTOR BASED ON M.M.F. CONSIDERATION

BY

M. NOURELDIN ABDEL-HAMID, Ph. D., M. Sc.

SUMMARY

It is shown that the equivalent circuit of the squirrel-cage induction motor can be deduced in an independent way by the consideration of the air — gap m.m.f. waves. Inspection of the m.m.f. balance equation of the machine will finally lead to its equivalent circuit with its constants expressed in terms of phase values.

The m.m.f. set up by a rotor cage placed in a sinusoidal rotating field is shown to be also sinusoidal with all harmonics of any practical importance completely suppressed by the cage winding.

LIST OF PRINCIPAL SYMBOLS

A	= Area of a pole = $\frac{\pi \cdot D \cdot L}{2 P}$	. . .	mt^2
B_g	= Air-Gap flux density.	. . .	Weber/ mt^2
D	= Stator bore		mts
$E_{b \max}$	= Maximum value of bar e.m.f.	. .	volts
$E_{s \max}$	= " " " stator e.m.f.	. .	"
e_b	= Instantaneous value of bar e.m.f.	. .	"
e_s	= " " " stator e.m.f.	. .	"
e_t	= " " " e.m.f. of one stator turn	"	"
f_1	= Supply frequency		cycles/sec.

$I_{b \max}$	= Maximum value of bar current . . .	amp
$I_{s \max}$	= " " " stator current . . .	"
i_{bm}	= Instantaneous value of the current of the m^{th} rotor bar	Amp
i_s	= Instantaneous value of stator current . . .	"
k_s	= Stator winding factor.	
k_{sk}	= Rotor skewing factor.	
K	= Constant	
L	= Axial length of the machine . . .	mts
L_g	= Corrected length of the air gap . . .	mts
N	= Number of rotor bars	
P	= Number of pole pairs	
R_1	= Stator resistance per phase	ohm
s	= per unit slip.	
t	= time	secs
v	= Speed	mts/sec.
W_s	= Number of Stator turns per phase . . .	
X_1	= Stator leakage reactance per phase . . .	ohms
Z'_2	= Rotor impedance per phase referred to the stator.	ohms
α	= Angle of lag of stator current behind stator e.m.f.	elect. radians
θ	= Angle locating arbitrary point in the air gap measured from the fixed axis "ab" on stator side (Fig. 1)	Mech. radians.
θ'	= Angle measured from the axis "ab"	Elect. Radians.
w_1	= Supply angular frequency	elect radians/sec.
μ	= Permeability of the air gap assumed equal to that of free space	
ϕ	= Angle of rotor impedance	radians.

(1) INTRODUCTION

The usual way to derive the equivalent circuit of the induction motor is to start off from the equivalent circuit of the transformer and modify it to account for the rotation of the motor. In this paper our aim is to proceed in an independent way.

The independent method of tackling the problem is to assume a sinusoidally distributed air gap flux density wave, and proceed to find expressions for the e.m.f.s. and m.m.f.s. of the stator and rotor. Since the cage rotor is dealt with here, its m.m.f. is obtained by adding up the individual bar m.m.f.s. for the whole cage. Finally writing down the m.m.f. balance equation of the machine, the equivalent circuit of the cage is obtained by inspection.

The parameters of the equivalent circuit are expressed in terms of phase values; in this connection it may be important to mention that the M.K.S. system of units is applied throughout this paper.

(2) AIR GAP M. M. F.

The air-gap flux density wave, assumed sinusoidal, can be expressed by:

$$B_g = B_{\max} \cdot \cos (P\theta - w_1 t) \quad . \quad . \quad . \quad (1)$$

The wave is rotating in the forward (positive) direction at the speed $w_1 / 2\pi \cdot p$ revolutions/sec.

From equation (1) and neglecting the iron losses of the machine, the magnetizing m.m.f. is:

$$(m.m.f.)_g = B_{\max} \cos (P\theta - w_1 t) \cdot L_g / \mu_g,$$

Putting $\mu_g = 4\pi \cdot 10^{-7}$, this last expression can then be modified to:

$$(m.m.f.)_g = B_{\max} \cdot \cos (P\theta - w_1 t) \cdot L_g \cdot \frac{10^7}{4\pi} \quad . \quad (2)$$

(3) STATOR E. M. F. AND M. M. F.

(3.1) STATOR E. M. F.

The e. m. f. induced in a single conductor of length L mts, cutting at right-angles a field B Webers/ mt^2 at a speed v mts/sec. is given by :

$$e = B \cdot L \cdot v \quad \text{volts.} \quad (3)$$

Considering one full-pitched turn of the stator whose sides lie at angles θ and $(\theta + \frac{\pi}{p})$ from the axis $a b$ (Fig 1), the e.m.f. induced in the turn due to the flux density wave given by equation (1) is :

$$e_t = - B_{\max} \cdot L \cdot \frac{\pi \cdot f_1 \cdot D}{p} \cdot \left[\cos (P\theta - w_1 t) - \cos \left\{ \left(\theta + \frac{\pi}{p} \right) P - w_1 t \right\} \right] \text{ volts.}$$

$$\therefore e_t = - 4 \cdot f_1 \cdot A \cdot B_{\max} \cdot \cos (P\theta - w_1 t)$$

Considering the whole W_s turns of the phase under consideration, the total e. m. f. induced in the stator phase is :

$$e_s = - 4 (W_s \cdot k_s B_{\max} \cdot f_1 \cdot A) \cdot \cos w_1 t \quad (4)$$

$$\text{or } e_s = - E_{s \max} \cdot \cos w_1 t \quad (5)$$

$$\text{where } E_{s \max} = 4 (W_s \cdot k_s \cdot B_{\max} \cdot f_1 \cdot A) \cdot \text{volts} \quad (6a)$$

$$\text{from which } B_{\max} = \frac{E_{s \max}}{4 \cdot W_s \cdot k_s \cdot f_1 \cdot A} \quad (6b)$$

The e.m.f. applied to the stator to compensate for e_s (equation 5) is equal and apposite to it i.e. equal to $E_{s \max} \cdot \cos w_1 t$. The stator current, assumed lagging by an angle α behind $-e_s$ is :

$$i_s = I_{s \max} \cdot \cos (w_1 t - \alpha) \quad (7)$$

The angle of lag α represents also the lag behind the applied stator voltage if the stator resistance and leakage reactance are neglected

(3.2) STATOR M. M. F.

In Fig 1 is shown the m.m.f. of one stator phase (which is considered the reference phase). The pulsating wave can be represented by a Fourier series including odd sine components only, the fundamental component being :

$$(m.m.f.)_{PH. "a"} = \frac{4}{\pi} \cdot \frac{W_s}{2p} \cdot i_s \cdot \frac{\sin \pi/6}{\pi/6} \sin \theta P \quad (8)$$

The factor $\frac{\sin \pi/6}{\pi/6}$ represents the effect of distribution of the stator winding. If we substitute for this factor by k_s to include the effect of any possible chording, and for i_s from equation (7) equation (8) will yield :

$$(m.m.f.)_{PH. "a"} = \left\{ \frac{4}{\pi} \cdot \frac{W_s \cdot k_s}{2p} \cdot I_{s \max} \right\} \cdot \cos (w_1 t - \alpha) \sin \theta p \quad (9a)$$

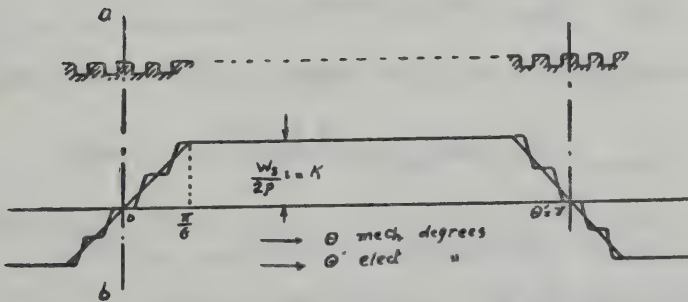


FIG. 1.—M.M.F. Wave of one Phase of a Polyphase Winding (slots/pole/phase = 4)

Similarly the m.m. fs of the other two stator phases are :

$$(m.m.f.)_{PH. "b"} = \left\{ \frac{4}{\pi} \cdot W_s \cdot \frac{k_s}{2p} \cdot I_{s \max} \right\} \cdot \cos \left(w_1 t - \alpha - \frac{2\pi}{3} \right) \cdot \sin \left(\theta p - \frac{2\pi}{3} \right) \quad (9b)$$

$$(m.m.f.)_{PH. "c"} = \left\{ \begin{aligned} & \left[\frac{4}{\pi} \cdot W_s \cdot \frac{k_s}{2p} \cdot I_{s \max} \right] \cdot \cos \\ & \left(w_1 t - \alpha - \frac{4\pi}{3} \right) \cdot \sin \left(\theta p - \frac{4\pi}{3} \right) \end{aligned} \right\} \quad (9c)$$

From the above expressions, the resultant stator m.m.f. is the sum of the phase m.m.f.s., thus :

$$(m.m.f.)_s = \frac{3}{\pi} \cdot \frac{W_s \cdot k_s}{P} \cdot I_{s \max} \cdot \sin (p\theta + \alpha - w_1 t) \quad (10)$$

(4) ROTOR REACTION

(4-1) E. M. F. and M. M. F. OF A ROTOR BAR

At any slip s , the air gap flux density wave (eq. (1)) rotates at a speed $\left(\frac{w_1}{P} \cdot s \right)$ mech. radians/sec. with respect to the rotor. Consequently, the air-gap flux density wave w.r.t. the rotor can be expressed by :

$$B_g = B_{\max} \cdot \cos (p\theta - s w_1 t) \quad (1a)$$

Referring back to eq. (3), the max e.m.f. of a rotor bar is :

$$\begin{aligned} E_{b \max} &= k_{sk} \cdot B_{\max} \cdot L \cdot \frac{s \cdot \pi \cdot f_1 D}{P} \quad \text{volts} \\ \therefore E_{b \max} &= 2 \cdot k_{sk} \cdot B_{\max} \cdot A \cdot s f_1 \quad \text{volts} \end{aligned} \quad (11)$$

Substituting, for $B_{\max}$ from eq. (6b), we get :

$$E_{b \max} = \frac{1}{2} \cdot \frac{s k_{sk}}{(W_s \cdot k_s)} \cdot E_{s \max} \quad (12)$$

The max bar current is : $E_{b \max} / |Z_b|$:

$$I_{b \max} = \frac{1}{2 |Z_b|} \cdot \frac{s k_{sk}}{(W_s \cdot k_s)} \cdot E_{s \max} \quad (13)$$

where Z_b represents the impedance of a rotor bar including the effect of the short-circuiting ring¹ at frequency sf_1 .

Considering the axis cd Fig. 2 of the mesh formed of the first and last rotor bar coinciding with the axis ab Fig. 1, and with the rotor bars numbered as in Fig 2, the current i_{bm} of the m^{th} rotor bar is given by :

$$i_{bm} = -I_{b \max} \cdot \cos \left[sw_1 t + \frac{2\pi p}{N} (m-1) + \frac{\pi p}{N} - \phi \right] \quad (14)$$

where ϕ is the angle of the impedance Z_b

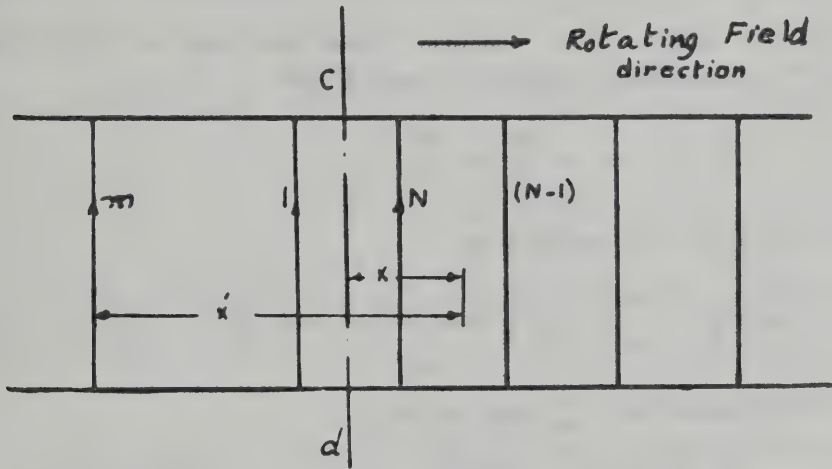


FIG. 2.—Numbering of Rotor Bars

Neglecting the m.m.f. drop in the rotor iron, the main circuit of the flux of any considered rotor bar will be two airgaps in series². Consequently, the m.m.f. of one rotor bar will be rectangular as in Fig. 3. The wave can be represented by a Fourier series which has odd sine harmonics only. The series is then of the form :

$$b_1 \cdot \sin X' + b_3 \sin 3 X' + \dots b_n \cdot \sin nX'$$

where X' is the mech. angle measured from the position of the m^{th} bar. The constant b_n can be proved to be given by :

$$b_n = \frac{i_{bm}}{n \pi}$$

and the m.m.f. of the m^{th} bar is:

$$(m.m.f.)_m = \sum_{n=1,3,5}^{n=\infty} \frac{i_{bm}}{\pi} \cdot \frac{1}{n} \cdot \sin n X' \quad (15)$$

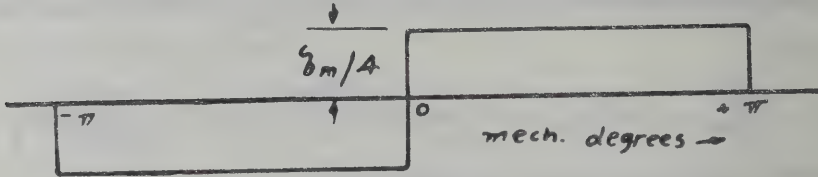


FIG. 3.—M.M.F. Wave of a Single Bar

If the angle X' is measured from the axis cd Fig 2 and called X , the relation between X' and X is:

$$X' = X + m \cdot \frac{2\pi}{N} - \frac{\pi}{N}$$

and Eq (15) can be modified to:

$$(m.m.f.)_m = \sum_{n=1,3,5}^{n=\infty} \frac{i_{bm}}{n \pi} \sin \left(n X + nm \cdot \frac{2\pi}{N} - \frac{n\pi}{N} \right) \quad (15a)$$

Substituting for i_{bm} from eq. 14:

$$(m.m.f.)_m = \sum_{n=1,3,5}^{n=\infty} \frac{-I_b \max}{n \pi} \cdot \cos \left[sw_1 t + \frac{2\pi P}{N} (m-1) \right] \left\{ \begin{array}{l} + \frac{\pi P}{N} - \phi \end{array} \right\} \cdot \sin \left(n X + nm \cdot \frac{2\pi}{N} - \frac{n\pi}{N} \right) \quad (16)$$

The $(m.m.f.)_{m,n}$ which is produced by the rotor bar m and of the order n , where n is an odd number, is a pulsating m.m.f. and can be resolved to a forward and backward rotating m.m.f.s, these are given by:

$$\text{Forward } (m.m.f.)_{m,n} = \frac{-I_b \max}{2 n \pi} \sin \left[n X + \frac{2\pi m}{N} \cdot (n - P) - \frac{\pi}{N} (n - P) + \phi - sw_1 t \right] \quad (17a)$$

$$\text{Backward (m.m.f.)}_{m,n} = \frac{-I_{b \max}}{2 n \pi} \sin \left[n X + \frac{2 \pi m}{N} \left\{ (n + P) - \frac{\pi}{N} (n + P) - \phi + s w_1 t \right\} \right] \quad (17b)$$

(4.2) M.M.F. OF A ROTOR CAGE

The resultant rotor m.m.f. in the forward direction is the sum of individual bar m.m.f.s given by equation (17a). Thus the rotor m.m.f. wave of order “n” in the forward direction of the rotor cage is:

$$\begin{aligned} (\text{m.m.f.})_{r,n, \text{for}} &= \sum_{m=1}^{m=N} \frac{-I_{b \max}}{2 n \pi} \cdot \sin \left[n X + \frac{2 \pi m}{N} (n - p) \right. \\ &\quad \left. - \frac{\pi}{N} \cdot (n - p) + \phi - s w_1 t \right] \\ &= \frac{-I_{b \max}}{2 n \pi} \sum_{m=1}^{m=N} \sin \left[\phi_1 + (n - p) \cdot \frac{\pi}{N} \cdot (2m - 1) \right] \\ &\quad \text{where } \phi_1 = (n X + \phi - s w_1 t) \\ &= \frac{-I_{b \max}}{2 n \pi} I M \sum_{m=1}^{m=N} \left[\phi_1 + (n - p) \cdot \frac{\pi}{N} \cdot (2m - 1) \right] \end{aligned}$$

where I M = imaginary part.

$$(\text{m.m.f.})_{r,n, \text{For}} = \frac{-I_{b \max} \cdot N}{2 n \pi} \cdot \frac{\sin [(n - p) \cdot \frac{\pi}{N}]}{N \cdot \sin [(n - p) \frac{\pi}{N}]} \left\{ \begin{array}{l} \sin [nX + \phi + (n - p) \pi - s w_1 t] \end{array} \right\} \quad (18a)$$

In a similar way and applying equation (17b), the backward (m.m.f.)_{r,n,Back} is:

$$(\text{m.m.f.})_{r,n, \text{back}} = \frac{-I_{b \max} \cdot N}{2 n \pi} \cdot \frac{\sin [(n + p) \cdot \frac{\pi}{N}]}{N \cdot \sin [(n + p) \frac{\pi}{N}]} \left\{ \begin{array}{l} \sin [nX - \phi + (n + p) \pi + s w_1 t] \end{array} \right\} \quad (18b)$$

Considering now the constants:

$$K_1 = \frac{\sin [(n - p) \pi]}{N \cdot \sin \left[(n - p) \frac{\pi}{N} \right]} \quad (19a)$$

and
$$K_2 = \frac{\sin [(n + p) \cdot \pi]}{N \cdot \sin \left[(n + p) \frac{\pi}{N} \right]} \quad (20a)$$

which appear in equations (18), we get the following points:

(a) The factor K_1 is generally equal to zero.

(b) In the case $\frac{(n - p)}{N} = u$, where u is an integer, K_1 will not be equal to zero but will have a value ± 1 according to whether $(uN - u)$ is even or odd.

(c) The case $\frac{n - p}{N} = u$ is hardly fulfilled except with $u = 0$ which leads to $n = p$. This means that the forward rotor (m.m.f.) is a fundamental wave.

(d) Consideration of K_2 (equation 20 a) shows that K_2 is generally zero except in the case $\frac{n + p}{N} = u$. This condition cannot be verified with low order of n . This means that the backward component of the whole cage will cancel out.

From the above, and putting $n = P$ and $K_1 = 1$ in equation (18 a) the resultant cage m.m.f. (= the forward wave) is:

$$(m.m.f.)_r = \frac{-I_b \max}{2 p \cdot \pi} \cdot k_{sk} \cdot \sin [p X + \phi - sw_1 t] \quad (21)$$

in which the factor k_{sk} is added to generalize it since the effect of skewing of rotor bars is clearly to decrease the (m.m.f.).

Substituting for $I_{b \max}$ in eq. (21) from eq. (13) we get:

$$(m.m.f.)_r = \frac{-E_s \max}{\frac{4.p.\pi}{N} \cdot s \cdot \frac{W_s \cdot k_s}{(k_{sk})^2}} \cdot \sin [P X + \phi - s w_1 t] \quad (22)$$

Since the rotor is rotating at a speed $\frac{(1-s)w_1}{P}$ mech. radians/sec,

then if X in eq. (22) is replaced by θ measured from the fixed axis $a b$ Fig. 1, equation (22) will be modified to:

$$(m.m.f.)_r = \frac{-E_s \max}{\frac{4.p.\pi}{N} \cdot s \cdot \frac{W_s \cdot k_s}{k_{sk}^2}} \sin [P \theta + \phi - w_1 t] \quad (23)$$

(5) M.M.F. BALANCE EQUATION

First, the final expression for the $(m.m.f.)_g$ is obtained by substituting for $B_{\max}$ in equations (2) from equation (6b), this leads to:

$$(m.m.f.)_g = \frac{E_s \max}{\frac{4 \pi}{10^7} \cdot \frac{4 \cdot W_s \cdot k_s \cdot f_1 \cdot A}{L_g}} \cos. (P \theta - w_1 t) \quad (24)$$

$$\text{Since: } (m.m.f.)_s + (m.m.f.)_r = (m.m.f.)_g \quad (25)$$

Equations (10), (23) and (24) will lead to the following (m.m.f) balance equation:

$$\begin{aligned} & \frac{3}{\pi} \frac{W_s \cdot k_s}{P} \cdot I_{s \max} \cdot \sin (P \theta + \alpha - w_1 t) + \\ & \frac{-E_s \max}{\frac{4 \cdot P \cdot \pi}{N} \cdot s \cdot \frac{W_s \cdot k_s}{k_{sk}^2}} \sin [P \theta + \phi - w_1 t] \\ & = \frac{E_s \max}{\frac{4 \pi}{10^7} \cdot \frac{4 \cdot W_s \cdot k_s \cdot f_1 \cdot A}{L_g}} \cos (P \theta - w_1 t) \end{aligned}$$

Since the above equation is valid for all values of θ and t , it can be simplified by choosing the value $\theta = 0$. Dividing all terms by $\frac{3}{\pi} \cdot \frac{W_s \cdot k_s}{P}$ the following expression will be obtained:

$$I_{s \max} \sin(w_1 t - \alpha) = \left[\frac{E_{s \max}}{\frac{|Z_b|}{s} \cdot N \cdot \left(\frac{1}{2} \cdot k_{sk}\right)^2} \sin(w_1 t - \phi) + \frac{E_{s \max}}{\frac{4 \cdot \pi}{10^7} \cdot \frac{12 \cdot (W_s \cdot k_s)^2 \cdot f_1 \cdot A}{L_g \cdot \pi \cdot P} \sin\left(w_1 t - \frac{\pi}{2}\right)} \right] \quad (26)$$

(6) EQUIVALENT CIRCUIT OF THE CAGE MOTOR

Considering the above equation, the equivalent circuit of the machine may be regarded as an e.m.f. $= E_{s \max} \sin(w_1 t)$ applied across two impedances in parallel. These impedances are the cage impedances referred to the stator side:

$$Z'_2 = \frac{|Z_b|}{s} \cdot \frac{3}{N} \cdot \frac{(W_s \cdot k_s)^2}{\left(\frac{1}{2} \cdot k_{sk}\right)^2} \cdot \frac{1}{\phi} \text{ ohms.} \quad (27)$$

and the magnetizing reactance:

$$X_m = \frac{4 \cdot \pi}{10^7} \cdot \frac{12 \cdot (W_s \cdot k_s)^2 \cdot f_1 \cdot A}{L_g \cdot \pi \cdot P} \cdot \frac{1}{\pi/2} \text{ ohms.} \quad (28)$$

Inspection of equation (27) leads to the following points:

1. The rotor impedance appears divided by the slip s .
2. The cage rotor may be regarded as an N phase winding, each phase has $(\frac{1}{2} k_{sk})$ effective turns.

Equation (28) represents a pure reactance since the iron losses of the machine have not been taken into account. If required, they can be represented by a high resistance in parallel with the above two

parallel impedances. Moreover, adding the stator resistance and leakage reactance in series gives the normal equivalent circuit, as shown, in Fig. 4.

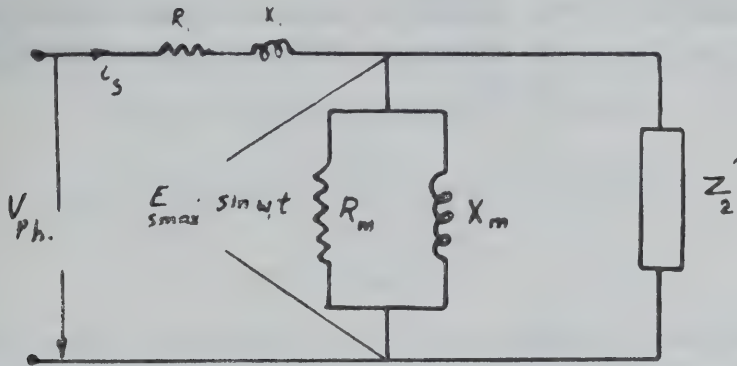


FIG. 4.—Equivalent Cct. of a Cage Motor

(7) CONCLUSIONS

The m.m.f. of a symmetrical cage winding has been shown to be almost purely sinusoidal. Only the m.m.f. having as many poles as the air-gap flux wave is built up by the cage, while all other harmonics of practical order are fully suppressed. Even those harmonics of very high order which might be developed could not have any noticeable effect since the impedance of the cage winding to such harmonics will be very high.

Also, equation (27) which shows the rotor impedance per phase referred to the stator side, confirms the method usually applied to treat the cage winding of the motor as having as many phases as the number of rotor bars. Each rotor bar in itself represents a phase, this means that each rotor phase has a number of turns = $\frac{1}{2}$.

The method of treatment applied in this paper, rather difficult as it may appear, allows the application of much of machine theory. It furthermore has the advantage of giving the final equivalent circuit constants in terms of phase parameters.

The method applied here will be a lot easier in the case of slip-ring induction motor, and is also applicable to a good many similar problems.

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APPLICATION OF SERIES CAPACITORS WITH PARALLEL SATURABLE INDUCTANCES IN A. C. TRANSMISSION LINES

BY

M. NOURELDIN ABDEL-HAMID, Ph. D., M. Sc.

INTRODUCTION

It is known that capacitors may be connected in series with A.C. transmission lines to compensate for the inductive drop in the line and, consequently, to increase the stability of the system. But an important consequence of doing so is the inevitable increase in the short-circuit current.

In this paper it is shown that the application of a combination of series capacitors and parallel saturable reactances serves the two purposes of compensating for the inductive drop under healthy conditions, and limiting short-circuit currents in case of short-circuit.

ASSUMPTIONS AND PRINCIPAL SYMBOLS

To simplify matters, we concern ourselves here with a single-phase transmission line, or what amounts to the same thing, one phase of a 3-ph. transmission line. The T.L. is considered short enough to neglect its charging current, and of negligible resistance.

The following symbols are used :

E = voltage at the sending end.

V_c = voltage across the series condenser.

V_{ca}, V_{cs} = condenser voltage in normal and short-circuit cases.

X_L = Inductive reactance of the transmission line including that of alternator, transformer ... etc.

- X_c = Capacitive reactance of the series condenser.
 X_R = Reactance of the saturable inductance connected in parallel with X_c
 I_{Ln}, I_{Ls} = Current in the transmission line under normal and short-circuit conditions, respectively.
 I_{cn}, I_{cs} = Current in the series condenser under normal and short-circuit conditions, respectively.
 I_{Rn}, I_{Rs} = Currents in the saturable inductance under normal and short-circuit conditions, respectively.
 k = I_{Ls} / I_{Ln}
 m = V_{cs} / V_{cn}

Also, it is assumed that the series condenser has zero dielectric losses and that the resistance of the saturable inductance is negligible.

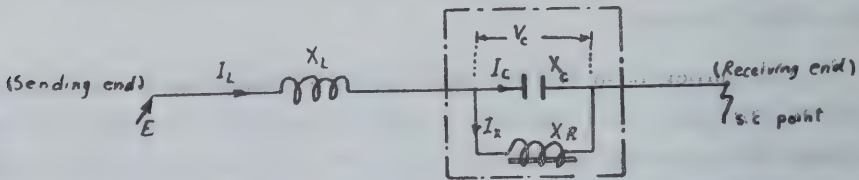


FIG. 1.—Representation of Short Transmission Line

THEORY

In case of no saturable inductance, X_c is designed to cancel the line inductive drop under healthy conditions, that is:

$$V_{cn} = I_{Ln} \cdot X_L = I_{Ln} \cdot X_c \quad (1)$$

In this case the voltage drop in the line will be only due to its resistance which is considered here negligible. With such arrangement, short circuit currents will be really high and can easily destroy the capacitor due to very high voltages across its terminals.

Connecting a saturable inductance across X_c , the combination is so designed to act capacitively on normal operation of the transmission line, and inductively at high values of V_c corresponding to the short-circuit case. Thus, on normal operation the combination reduces the

effective inductive drop of the T.L. and adds to the stability of the system, and in the s.c. case it increases the effective inductance, and thus helps to limit the short-circuit currents and reduces the capacity of the required circuit-breakers.

To understand the theory of the method we consider Fig. 2 showing voltage / current characteristics of X_c , X_R and the combination of both in parallel. Curve (a) of the capacitor is linear, and curve (b) of the saturable inductance is linear up to a certain voltage representing the unsaturated operation then it curves quickly into the saturated region. As regards curve (c) of the combination it is concluded by subtracting for any voltage the magnitudes of the two currents obtained from curves (a) and (b). It is seen that up to the voltage V_{co} the combination is effectively capacitive with a maximum current $I_{c \max}$, while for higher voltages it is inductive. The normal

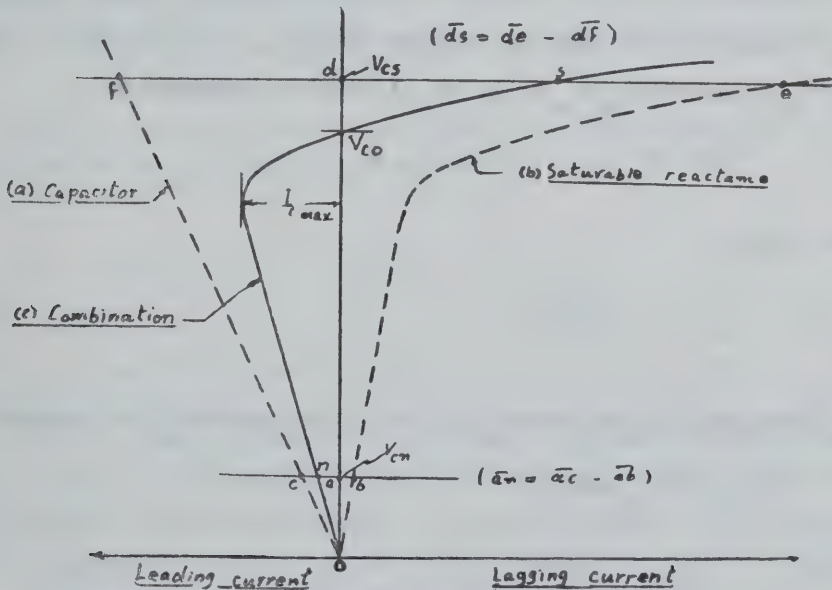


FIG. 2.—Volt./Current Characteristics of Capacitor, Saturable Reactance, and Their Parallel Combination

voltage across the combination should be chosen much lower than V_{co} , i.e. $V_{cn} < V_{co}$, while in the case of short-circuit $V_{cs} > V_{co}$, the operating points being respectively n and s. These two points

determine the ratios of voltages and currents in the normal and s.c. cases, *i.e.* m and k , and therefore they are the decisive factors in designing the saturable inductance.

Noting that all symbols in the following equations represent magnitudes, the currents in the transmission line in the normal and short-circuit cases are:

$$I_{Ln} = I_{cn} - I_{Rn} \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and
$$I_{Ls} = I_{Rs} - I_{cs} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In the short-circuit case:

$$E = V_{cs} + I_{Ls} \cdot X_L \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Assuming that the normal voltage across the series capacitor V_{cn} is equal to the reactive voltage drop in the line, and calling $V_{cs}/V_{cn} = m$ and $I_{Ls}/I_{Ln} = k$, eq. (4) will be modified to:

$$E = m I_{Ln} \cdot X_L + k I_{Ln} \cdot X_L$$

from which:

$$I_{Ln} = \frac{E}{(m + k) \cdot X_L} \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

From eq. (2), the normal current in the saturable inductance is:

$$\begin{aligned} I_{Rn} &= I_{cn} - I_{Ln} \\ &= I_{Ln} \left(\frac{X_L}{X_c} - 1 \right) \end{aligned}$$

and from eq. (5):

$$\therefore I_{Rn} = E \frac{\frac{X_L}{X_c} - 1}{(m + k) \cdot X_L} \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Also, from eq. (3) representing the short-circuit case :

$$\begin{aligned} I_{Rs} &= I_{Ls} + I_{cs} \\ &= k. I_{Ln} + V_{cs}/X_c \\ &= k. I_{Ln} + m I_{Ln} \cdot \frac{X_L}{X_c} \\ &= I_{Ln} \left(m \cdot \frac{X_L}{X_c} + k \right) \end{aligned}$$

and from eq. (5), the current in the saturable inductance in the short-circuit case is therefore :

$$I_{Rs} = E \frac{m \cdot \frac{X_L}{X_c} + k}{(m + k) \cdot X_L} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The voltage in the normal and s.c. cases across either the capacitor or the saturable inductance are :

$$V_{cn} = I_{Ln} \cdot X_L = \frac{E}{m + k} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$\text{and} \quad V_{cs} = m \cdot V_{cn} = \frac{E \cdot m}{m + k} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

In the design office, the relations expressed by the above Eqs. 1 and 6 to 9, should be helpful in carrying out the design of the appropriate saturable inductance for certain chosen ratios k and m which determine the s.c. current and maximum capacitor voltage in terms of the normal corresponding values.

CONCLUSIONS

It has been shown that the simple connection of a capacitor in series with the transmission line can be fully advantageous on normal operation. But in the case of short-circuit, such method may result

in excessive s.c. currents and voltage across the capacitor leading to heavy and expensive switchgear, and destruction of the capacitor.

By connecting a saturable inductance of appropriate design across the terminals of the capacitor, the double aim of improving the transmission of power under healthy conditions as well as limiting the line current and capacitor voltage in the case of short-circuit can be served efficiently.

The design of saturable inductances is no longer a laborious problem in the light of increasing knowledge about magnetic materials and winding techniques¹. As is well known, d.c. controlled saturable inductances are nowadays increasingly used in modern practice.

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FILTERS POSSESSING APPROXIMATE GAUSSIAN TRANSFER FUNCTION

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SUMMARY

The simplest among the several two terminal-pair networks which possess the transfer-function $1/(1 + j \tau \omega)^n$ is the low-pass ladder network shown in Figures 2e, and 7. The general design formulae of this network are given. The band-pass ladder is also derived for radio-frequency pulses.

INTRODUCTION

The production of Gaussian error-curve pulses by means of networks having a Gaussian transfer-function is well-known [1,2,3,4,5,6]. A transfer-function $Ke^{-(\frac{\omega}{\omega_0})^2}$ having zero phase is physically not realizable [2, 3, 6]. On the other hand the physically realizable transfer function $K'/(1 + j \tau \omega)^n$ can approximate in amplitude to any desired degree the function $Ke^{-(\frac{\omega}{\omega_0})^2}$ [1, 2, 3, 6]. The phase, however, is not zero; it grows with n and diverges for infinitely large n [2, 3, 6]. Figure 1 shows the relation between the transfer function and the normalized frequency for different values of n .

Such transfer function can be realized by a cascade connection of n identical amplifier stages, each having as plate-load a resistance

and a capacitance in parallel (Figure 2a). n should be as large as possible for the approximation to the Gaussian error curve to be reasonable. Thus a large number of amplifier stages is needed, and this solution is too complicated and expensive for the purpose.

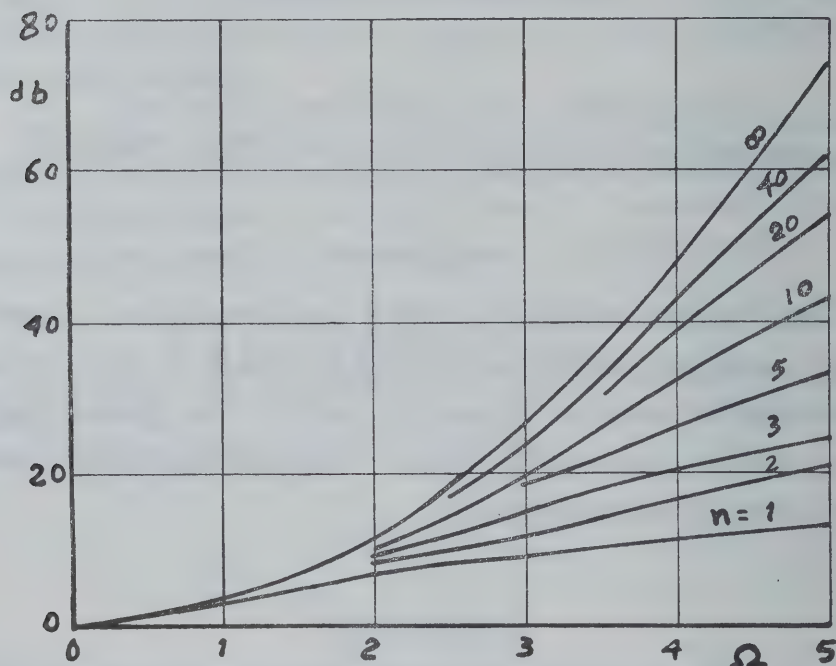


FIG. 1

The passive networks of Figure 2 b, c and d are constant-resistance networks which have the desired transfer function [6, 9] as is shown in the appendix. They are essentially simpler and cheaper than the multistage amplifier. They need, however, too many circuit-elements. The networks of Figure 2 b and c need $(3n + 1)$ elements; the network of Figure 1 d needs $(4n + 1)$ elements.

The low-pass ladder of Figures 2 e, 5 and 7 can be designed to possess the required transfer function. It possesses, like the networks of Figure 2 b, c and d the advantage of being passive. Further it is much simpler than the mentioned networks. It has only $(n + 1)$ circuit-elements, and is thus canonic.

Such a ladder has already been calculated and used [1], but no general theory of the network was developed, and only special cases with small values of n were treated. The general theory developed here readily permits the calculation of more complicated ladders and thus allows better approximations to the Gaussian error-curve.

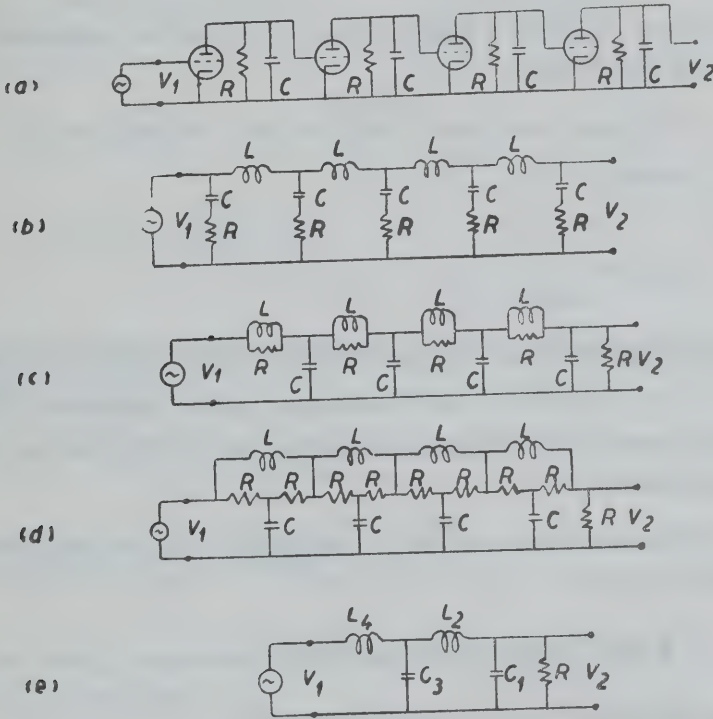


FIG. 2

For pulses of high-frequency carrier a band-pass ladder will also be derived (Figure 9).

THE LOW-PASS LADDER. CONSTANT VOLTAGE DRIVE.

When the drive is a constant voltage generator the transfer function is taken to be the open-circuit voltage transfer-ratio:

$$T = \left[\frac{V_2}{V_1} \right]_{I_2 = 0} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

(Figure 3a). This is the case of a cathode follower stage drive.

The positive value being taken (otherwise the input or the output terminals can be crossed). It follows:

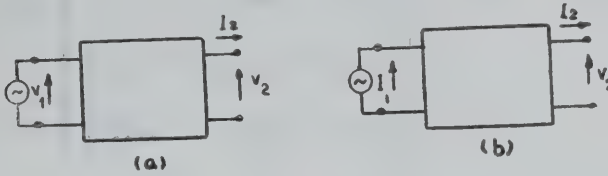


FIG. 3

$$T(p) = \frac{1}{(1 + \tau p)^n} \quad . \quad . \quad . \quad . \quad . \quad (8)$$

A two terminal-pair network (Figure 3a) has generally the relations:

$$V_1 = A'_{11} V_2 + A'_{12} I_2 \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$I_1 = A'_{21} V_2 + A'_{22} I_2 \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The voltage transfer-ratio is the voltage ratio when the output current I_2 is zero, thus:

$$T = \left[\frac{V_2}{V_1} \right]_{I_2 = 0} = \frac{1}{A'_{11}} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Let the relations (9) and (10) refer to network I in Figure 4a, composed of network II terminated in pure resistance R at its output side. Let network II have the chain matrix:

$$\| A \| = \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

It follows that the network I has the chain matrix:

$$\left\| \begin{array}{cc} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{array} \right\| = \left\| \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right\| \cdot \left\| \begin{array}{cc} 1 & 0 \\ \frac{1}{R} & 1 \end{array} \right\| \quad (13)$$

$$= \left\| \begin{array}{cc} A_{11} + \frac{1}{R} A_{12} & A_{12} \\ A_{21} + \frac{1}{R} A_{22} & A_{22} \end{array} \right\|$$

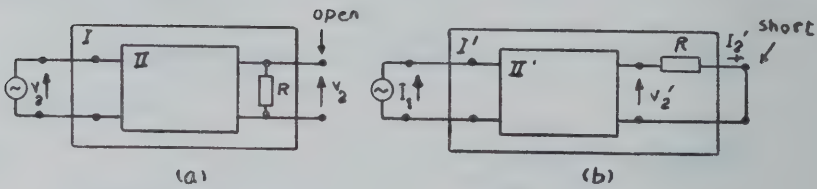


FIG. 4

The transfer-ratio of network I is then given by :

$$T = \frac{1}{A'_{11}} = 1 / \left[A_{11} + \frac{1}{R} A_{12} \right] \quad (14)$$

Network I has thus the required transfer-ratio (8), when :

$$\left(A_{11} + \frac{1}{R} A_{12} \right) = (1 + \tau p)^n \quad (15)$$

Now let network II be purely reactive. In this case A_{11} and $\frac{1}{R} A_{12}$ can be separately determined from (15) as follows. For $p = j\omega =$ pure imaginary, A_{11} is a pure real quantity and A_{12} is a pure imaginary quantity for reactive networks. Thus since (15) is real for real p , A_{11} is the even part and $\frac{1}{R} A_{12}$ is the odd part of $(1 + \tau p)^n$. Thus for $p = j\omega =$ pure imaginary :

$$A_{11} = \text{Re} \left((1 + \tau p)^n \right) \quad (16)$$

$$\frac{1}{R} A_{12} = \text{Im} \left((1 + \tau p)^n \right) \quad (17)$$

where Re and Im mean the real and imaginary parts. Let:

$$q = \tau p. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

$$\text{Hence } A_{11} = \frac{1}{2} \left((1 + q)^n + (1 - q)^n \right) \quad . \quad . \quad . \quad (19)$$

$$A_{11} = a_0 + a_2 q^2 + a_4 q^4 + \dots + \frac{a_{n-1} q^{n-1}}{a_n q^n} \quad \begin{matrix} n = \text{odd} \\ n = \text{odd} \end{matrix} \quad (20)$$

$$\frac{1}{R} A_{12} = \frac{1}{2} \left((1 + q)^n - (1 - q)^n \right) \quad . \quad . \quad . \quad (21)$$

$$\frac{1}{R} A_{12} = a_1 q + a_3 q^3 + a_5 q^5 + \dots + \frac{a_n q^n}{a_{n-1} q^{n-1}} \quad \begin{matrix} n = \text{odd} \\ n = \text{even} \end{matrix} \quad (22)$$

$$\text{where } a_v = \begin{pmatrix} v \\ n \end{pmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

From the reciprocity relation for network II follows:

$$| A | = (A_{11} A_{22} - A_{12} A_{21}) = 1 \quad . \quad . \quad . \quad (24)$$

Hence the following relations for network II:

$$I_1 = Y_{11} V_1 - Y_{12} V_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

$$I_2 = Y_{21} V_1 - Y_{22} V_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

where the short -circuit admittances are given by :

$$Y_{11} = \frac{A_{22}}{A_{12}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

$$Y_{12} = Y_{21} = \frac{1}{A_{12}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

$$Y_{22} = \frac{A_{11}}{A_{12}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

Substituting (20) and (22) in (29) one gets:

$$Y_{22} = R \frac{a_0 + a_2 q^2 + a_4 q^4 + \dots + a_n q^n}{a_1 q + a_3 q^3 + a_5 q^5 + \dots + a_n q^n} \quad \begin{matrix} n = \text{odd} \\ n = \text{even} \end{matrix} \quad (30)$$

Y_{22} will now be developed into a continuous fraction of the Cauer—Guillemin type [7, 10, 11].

For $n = \text{odd}$:

$$Y_{22} = \frac{1}{R} \left(\frac{1}{\alpha_1 q} + \frac{1}{\alpha_2 q} + \frac{1}{\alpha_3 q} + \dots + \frac{1}{\alpha_{n-1} q} + \frac{1}{\alpha_n q} \right) \quad (31)$$

$$= \left(\frac{1}{\beta_1 p} + \frac{1}{\beta_2 p} + \frac{1}{\beta_3 p} + \dots + \frac{1}{\beta_{n-1} p} + \frac{1}{\beta_n p} \right) \quad (32)$$

where $\beta_v = \tau \alpha_v R$ (33)

for $v = \text{odd}$, and:

$$\beta_v = \frac{\tau \alpha_v}{R} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

for $v = \text{even}$.

For $n = \text{even}$:

$$Y_{22} = \frac{1}{R} \left(\alpha_1 q + \frac{1}{\alpha_2 q} + \frac{1}{\alpha_3 q} + \dots + \frac{1}{\alpha_{n-1} q} + \frac{1}{\alpha_n q} \right) \quad (34)$$

$$= \left(\beta_1 p + \frac{1}{\beta_2 p} + \frac{1}{\beta_3 p} + \dots + \frac{1}{\beta_{n-1} p} + \frac{1}{\beta_n p} \right) \quad (35)$$

where $\beta_v = \frac{\tau \alpha_v}{R}$ (36)

for $v = \text{odd}$, and:

$$\beta_v = \tau \alpha_v R \quad (37)$$

for $v = \text{even}$.

The continuous fraction development can be made either by long division or by a method given by Cauer [7]. The last one will be used here for it gives the result explicitly. Although the order of the a -coefficients in (30) is the reverse of Cauer's, the results there are applicable here because $a_v = a_{n+1-v}$. Let the determinants Δ_v ($v = 1$ to n) be defined as follows:

$$\Delta_{-1} = \frac{1}{a_0}, \quad \Delta_0 = 1$$

$$\begin{aligned} \Delta_1 &= \begin{vmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & \dots & 0 \end{vmatrix} \\ \Delta_2 &= \begin{vmatrix} a_3 & a_2 & a_1 & a_0 & 0 & 0 & \dots & 0 \end{vmatrix} \\ \Delta_3 &= \begin{vmatrix} a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \dots & 0 \end{vmatrix} \\ \Delta_4 &= \begin{vmatrix} a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & \dots & 0 \end{vmatrix} \\ \Delta_5 &= \begin{vmatrix} a_9 & a_8 & a_7 & a_6 & a_5 & a_4 & \dots & 0 \end{vmatrix} \\ \Delta_6 &= \begin{vmatrix} a_{11} & a_{10} & a_9 & a_8 & a_7 & a_6 & \dots & 0 \end{vmatrix} \\ &\vdots \\ \Delta_n &= \begin{vmatrix} a_{2n-1} & a_{2n-2} & \dots & a_n \end{vmatrix} \end{aligned} \quad (38)$$

The explicit relation between α_v and $a_0, a_1, \dots, a_n$ is [7]:

$$\alpha_v = \frac{\Delta_{v-1}^2}{\Delta_{v-2} \Delta_v}$$

where the Δ 's are defined by (38) and the a 's by (23).

The network can now be recognized as a ladder network having series inductances and parallel capacitances as shown in Figure 5. The total number of circuit-elements (R n t included) is n . The inductance or capacitance of the circuit-element is β_v . The input element β_n is always an inductance.

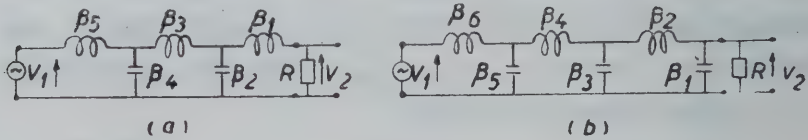


FIG. 5

Usually τ will not be specified, but rather the frequency f_1 at which the amplitude at the output is 3 decibels lower than the maximum at zero frequency. Thus:

$$\left(1 + (2\pi f_1 \tau)^2\right)^{1/2 n} = \sqrt{2}$$

$$\text{therefore } \tau = \sqrt{2^{1/n} - 1} \cdot \frac{1}{2\pi f_1} \quad (40)$$

This can be approximated for large n as:

$$\tau = \sqrt{\frac{1}{n} \ln 2} \cdot \frac{1}{2\pi f_1} = \frac{0.832}{\sqrt{n}} \cdot \frac{1}{2\pi f_1} \quad (41)$$

The error in (41) is 4.6% for $n = 4$ and 2% for $n = 9$.

Now the values of the circuit elements of the ladder Figure (5) are given by

$$\beta_v = \tau \alpha_v R \quad (42)$$

for the series inductances, and by:

$$\beta_v = \tau \alpha_v \frac{1}{R} \quad (43)$$

for the shunt capacitances. The input circuit-element β_n is always an inductance. In (42) and (43) τ is given by (40) and α_v by (39). Otherwise α_v is obtained from the following table or from Figure 6.

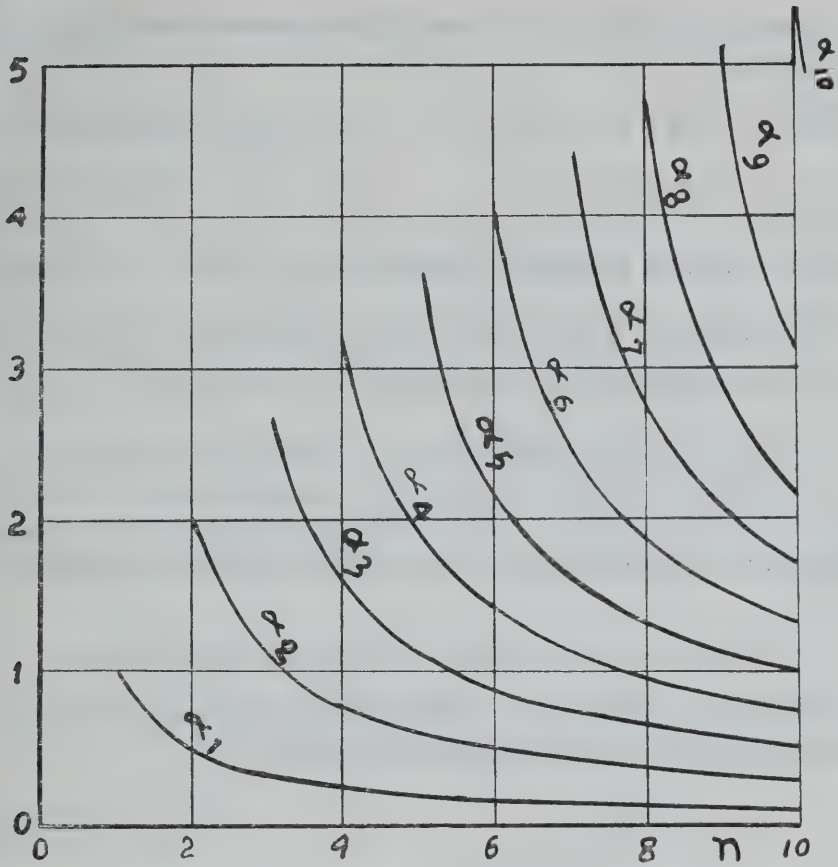


FIG. 6

n	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}
1	1.000									
2	0.500	2.000								
3	0.333	1.125	2.667							
4	0.250	0.800	1.562	3.200						
5	0.200	0.625	1.143	1.914	3.657					
6	0.167	0.514	0.911	1.422	2.15	4.064				
8	0.125	0.381	0.656	0.970	1.354	1.877	2.724	4.773		
10	0.100	0.303	0.516	0.746	1.006	1.312	1.701	2.186	3.116	5.456

For example for $n = 5$ (Figure 5a) the values of the circuit-elements are as follows :

$$\begin{aligned} L_1 = \beta_1 &= 0.200 k R / (2 \pi f_1), & C_2 = \beta_2 &= 0.625 k / (R 2 \pi f_1), \\ L_1 = \beta_3 &= 1.143 k R / (2 \pi f_1), & C_4 = \beta_4 &= 1.914 k / (R 2 \pi f_1), \\ L_5 = \beta_5 &= 3.656 k R / (2 \pi f_1), & \text{where } k &= \sqrt{2^{1/2} - 1} = 0.387. \end{aligned}$$

The values of α_v can be evaluated as functions of n by calculating the corresponding Δ 's as functions of n . Thus one gets :

$$\alpha_1 = \frac{1}{n}, \alpha_2 = \frac{(n^2 - 1)}{3n}, \alpha_3 = \frac{5(n^2 - 1)}{n(n^2 - 4)}, \text{ etc.} \quad (44)$$

Such general expressions are, however, difficult to obtain for large v .

THE LOW-PASS LADDER. CONSTANT-CURRENT DRIVE.

When the driver is a constant-current generator the transfer function is taken to be the transfer-impedance :

$$Z_T = \left[\frac{V_2}{I_1} \right]_{I_2 = 0} \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

(Fig 3b). This is the case of drive from a pentode stage.

The necessary and sufficient conditions for a function to be the transfer-impedance of a physically realizable network having a finite number of circuit-elements are the following :

(1) $Z_T(p)$ should be real for p real.

(2) $Z_T(p) = \frac{P(p)}{Q(p)}$, where $P(p)$ is a polynomial and $Q(p)$ is a Hurwitz polynomial in p .

Should these conditions be satisfied, then the function can be the transfer-impedance of a physically realizable fourpole. This fourpole can further always be transformerless (while placing no further restrictions on the function).

The transfer-impedance :

$$Z_T = \left[\frac{V_2}{I_1} \right]_{I_2=0} = \frac{K}{(1 + \tau p)^n} \quad . \quad . \quad . \quad (46)$$

where K and τ are positive real constants, satisfies the mentioned conditions and is thus realizable. The pertinent fourpole can be synthesized in a manner similar to the synthesis performed in the last section. However, use will be made here of the principles of duality and reciprocity together with the results of the last section.

(i) Let the two fourpoles II and II' of Fig 4 be dual, the duality being taken with respect to the load resistance R . Obviously the open-circuit voltage transfer-ratio of one is equal to the short-circuit current transfer-ratio of the other.

Thus :

$$T_v = \left[\frac{V_2}{V_1} \right] = \left[\frac{I_2'}{I_1} \right] = T_i' \quad . \quad . \quad . \quad (47)$$

But since $I_2' = \frac{1}{R} V_2'$

it follows that the transfer impedance of the second fourpole is given by :

$$Z_T' = \frac{V_2'}{I_1'} = R \cdot \frac{V_2}{V_1} = R \cdot T_v \quad . \quad . \quad . \quad (48)$$

The fourpole which is dual to the one obtained above (fig. 5) possesses therefore the required transfer-impedance, on condition that

$$R = K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49)$$



FIG. 7

One gets a band-pass ladder as in Fig 9a having as parallel elements parallel resonant circuits and as series elements series resonant circuits. The series resonant circuits have:

$$L'_v = \sqrt{2^{1/n} - 1} \cdot \frac{R}{2\pi(f'_1 - f'_{-1})} \cdot \alpha_v \quad (61)$$

$$C'_v = \frac{1}{4\pi^2 f_o'^2 L'_v} \quad (62)$$

and the parallel resonant—circuits:

$$C'_v = \sqrt{2^{1/n} - 1} \cdot \frac{1}{2\pi R (f'_1 - f'_{-1})} \cdot \alpha_v \quad (63)$$

$$L'_v = \frac{1}{4\pi^2 f_o'^2 C'_v} \quad (64)$$

where α_v is given by (39) and the table in page 435.

Corresponding to the three previous low-pass ladders one gets three band-pass ladders respectively (Fig 9).

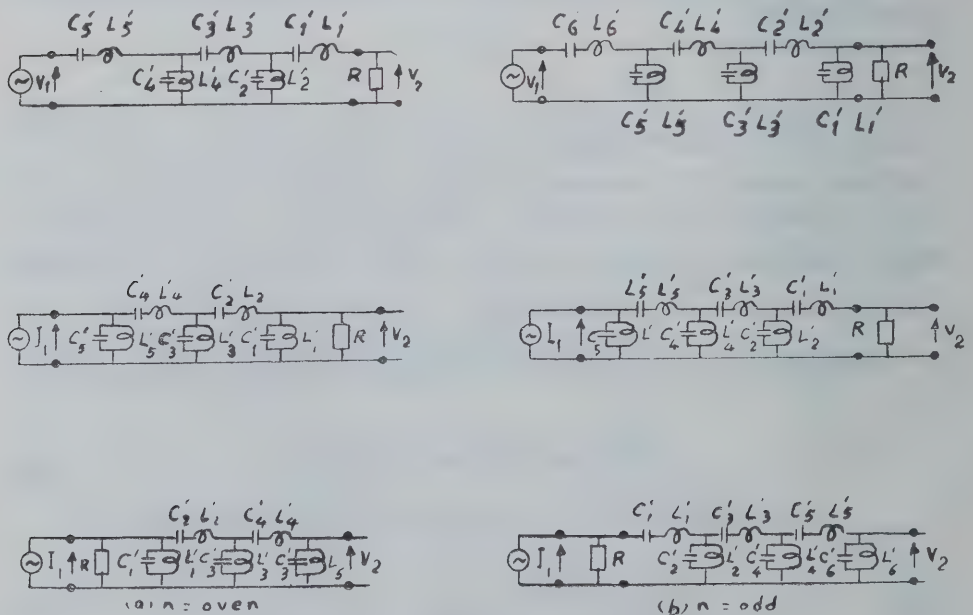


Fig. 9

APPENDIX

Constant-Resistance Networks

Having Transfer-Ratios $\frac{1}{(1 + j \tau \omega)^n}$

(a) Referring to Figure 10 a, let ;

$$R_1 = R_2 = R$$

and $Z_1 Z_2 = R^2$.

Then the input impedance is :

$$Z = R .$$

The output terminals will be taken across R_2 . The transfer-ratio is then :

$$T = \frac{V_2}{V_1} = 1 / \left(1 + \frac{Z_2}{R} \right) .$$

Let $Z_1 = \frac{1}{j\omega C}$,

$$Z_2 = j\omega L ,$$

where $\frac{L}{C} = R^2$.

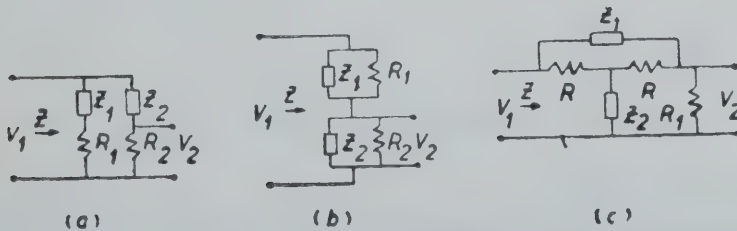


FIG. 10

In this case :

$$T = 1 / \left(1 + j \frac{\omega L}{R} \right) .$$

Because the input impedance is:

$$Z = R ,$$

such a network can substitute R_2 ; this can be further repeated. For n such networks (Figure 2b):

$$T = 1/(1 + j \tau \omega)^n,$$

where $\tau = \frac{L}{R} .$

(b) Referring to Figure 10 b , let :

$$R_1 = R_2 = R$$

and $Z_1 Z_2 = R^2 .$

Then the input impedance is:

$$Z = R .$$

The output terminals will be taken across R_2 . The transfer-ratio is then :

$$T = \frac{V_2}{V_1} = 1 / \left(1 + \frac{R}{Z_2} \right)$$

Let $Z_1 = j \omega L ,$

$$Z_2 = \frac{1}{j \omega C} ,$$

where $\frac{L}{C} = R^2 ,$

then $T = 1/(1 + j \omega C R).$

Because $Z = R ,$

a complete network can substitute R_2 ; this may be repeated further. For n such network (Fig 2 c):

$$T = 1/(1 + j \omega C R)^n = 1/(1 + j \tau \omega)^n$$

where $\tau = C R .$

(c) The network of Fig 10 c. where :

$$Z_1 Z_2 = R^2$$

has [9] (without R_1) the image impedance :

$$Z_1 = R$$

and the image transfer-ratio :

$$1 / \left(1 + \frac{Z_1}{R} \right) .$$

Adding $R_1 = R$

as in Fig 10 c one gets the input impedance :

$$Z = R$$

and the transfer-ratio :

$$T = 1 / \left(1 + \frac{Z_1}{R} \right) .$$

Let $Z_1 = j \omega L$,

$$Z_2 = \frac{1}{j \omega C} ,$$

where $\frac{L}{C} = R^2$,

then $T = 1 / \left(1 + j \frac{\omega L}{R} \right) .$

Now R_1 may be substituted for by a chain of such networks.

For n networks (Fig 2 d)

$$T = 1 / (1 + j \tau \omega)^n$$

where $\tau = \frac{\omega L}{R} .$

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